APPROXIMATING L^2-INVARIANTS AND HOMOLOGY GROWTH

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Abstract. In this paper we consider the asymptotic behavior of invariants such as Betti numbers, minimal numbers of generators of singular homology, the order of the torsion subgroup of singular homology, and torsion invariants. We will show that all these vanish in the limit if the CW-complex under consideration fibers in a specific way. In particular we will show that all these vanish in the limit if one considers an aspherical closed manifold which admits a non-trivial S^1-action or whose fundamental group contains an infinite normal elementary amenable subgroup. By considering classifying spaces we also get results for groups.

0. Introduction

In this paper we consider the asymptotic behavior of invariants such as Betti numbers, minimal numbers of generator of singular homology, the order of the torsion subgroup of singular homology, and torsion invariants. We will use the following setup.

Setup 0.1. Let G be a group together with an inverse system $\{G_i \mid i \in I\}$ of normal subgroups of G directed by inclusion over the directed set I such that $[G : G_i]$ is finite for all $i \in I$ and $\bigcap_{i \in I} G_i = \{1\}$. Let K be a field.

We want to study for a G-covering $\overline{X} \to X$ of a connected finite CW-complex X the nets

$$\left(\frac{b_n(G_i\backslash X;K)}{[G:G_i]}\right)_{i\in I}, \left(\frac{\ln\left(\left|\operatorname{tors}(H_n(G_i\backslash\overline{X}))\right|\right)}{[G:G_i]}\right)_{i\in I}, \left(\frac{\mathrm{d}(H_n(G_i\backslash\overline{X}))}{[G:G_i]}\right)_{i\in I},$$

$$\left(\frac{\mathrm{d}(G_i)}{[G:G_i]}\right)_{i\in I}, \left(\frac{\rho^{(2)}(G_i\backslash\overline{X};\mathcal{N}(\{1\}))}{[G:G_i]}\right)_{i\in I}, \quad\text{and}\quad \left(\frac{\rho^{\mathbb{Z}}(G_i\backslash\overline{X})}{[G:G_i]}\right)_{i\in I},$$

where d denotes the minimal number of generators, $b_n(G_i\backslash X;K)$ the nth Betti number with coefficients in the field K, $\operatorname{tors}(H_n(G_i\backslash\overline{X}))$ the torsion subgroup of the abelian group given by the nth singular homology with integer coefficients, $\rho^{(2)}(G_i\backslash\overline{X};\mathcal{N}(\{1\}))$ the L^2-torsion (with respect to the trivial group) and $\rho^{\mathbb{Z}}(G_i\backslash\overline{X})$ the integral torsion.

Of particular interest is the sequence $\frac{\ln\left(\left|\operatorname{tors}(H_n(G_i\backslash\overline{X}))\right|\right)}{[G:G_i]}$ for which most of the known results are restricted to infinite cyclic fundamental groups (see for instance [4, 29, 30]). The other sequences have already been studied intensively in the literature, see for instance [1, 2, 3, 6, 7, 8, 10, 16, 17, 18, 20, 24, 28].

We will concentrate on cases, where we can show that these sequences converge to zero, for instance, when X is a closed aspherical manifold, $G = \pi_1(X)$, $\Phi = \mathrm{id}_G$, $\overline{X} = \widetilde{X}$ and we assume either that X carries a non-trivial S^1-action or G contains

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a non-trivial elementary amenable normal subgroup (see Corollary 1.13). Thus we can prove in this special case the Approximation Conjectures of Subsection 1.4 which present one of the basic motivation for this paper.

If one considers $\overline{X} = EG$, one obtains statements about the group G itself as explained in Subsection 1.5.

We briefly discuss the related notions of rank gradient and cost in Subsection 1.6.

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1. Statement of results

1.1. Main result. Next we state our main technical result. All other results are essentially consequences of it. Explanations follow below.

Theorem 1.1 (Fibrations)**.** *Let $F \xrightarrow{j} X \xrightarrow{f} B$ be a fibration of connected CW-complexes. Consider a homomorphism $\phi\colon \pi_1(X) \to G$. Let $p\colon \overline{X} \to X$ be the associated G-covering. Let $G_1(F) \subseteq \pi_1(F)$ be Gottlieb's subgroup of the fundamental group of F (see Definition 1.6.) Suppose that the image of $G_1(F)$ under the composite $\phi \circ \pi_1(j)\colon \pi_1(F) \to G$ is infinite.*

If d is a natural number such that the $(d+1)$-skeleton of X is finite, then:

1. *We get for all $n \leq d$*
$$\lim_{i \in I} \frac{\mathrm{d}\big(H_n(G_i\backslash\overline{X})\big)}{[G : G_i]} = 0;$$
2. *We get for all $n \leq d$*
$$\lim_{i \in I} \frac{\ln\left(\big|\mathrm{tors}\big(H_n(G_i\backslash\overline{X})\big)\big|\right)}{[G : G_i]} = 0;$$
3. *We get for all $n \leq d$*
$$b_n^{(2)}\big(\overline{X}; \mathcal{N}(G)\big) = \lim_{i \to \infty} \frac{b_n(G_i\backslash X; K)}{[G : G_i]} = 0;$$
4. *Suppose that X is a connected finite CW-complex. Then*
$$\lim_{i \in I} \frac{\rho^{(2)}\big(G_i\backslash\overline{X}; \mathcal{N}(\{1\})\big)}{[G : G_i]} = \lim_{i \in I} \frac{\rho^{\mathbb{Z}}\big(G_i\backslash\overline{X}\big)}{[G : G_i]} = 0;$$
5. *Suppose that both F and B are connected finite CW-complexes and that $\rho^{(2)}(\overline{F}; \mathcal{N}(H)) = 0$, where H is the image of the composite $\pi_1(F) \xrightarrow{\pi_1(j)} \pi_1(E) \xrightarrow{\phi} G$ and $\overline{F}$ is the covering associated to the induced epimorphism $\pi_1(F) \to H$. Then the L^2-torsion $\rho^{(2)}\big(\overline{X}; \mathcal{N}(G)\big)$ is defined and satisfies*
$$\rho^{(2)}\big(\overline{X}; \mathcal{N}(G)\big) = 0.$$

Remark 1.2. We emphasize that we do *not* require in Theorem 1.1 that the fibre transport of the fibration $f\colon X \to B$ is trivial or that F, B or $G_1(F)$ satisfy any finiteness assumptions except in assertion (5).

Remark 1.3. Some of the assertions appearing in Theorem 1.1 imply one another. For instance , we have the following facts which are all consequences of the Universal Coefficients Theorem and Lemma 2.13

If we have $\lim_{i \in I} \frac{\mathrm{d}(H_n(G_i\backslash\overline{X}))}{[G:G_i]} = 0$ for all $n \leq d$, then we get for all $n \leq d$
$$\lim_{i \in I} \frac{b_n(G_i\backslash\overline{X}; K)}{[G : G_i]} = 0.$$

If $n \le d$ and we have both $\lim_{i\in I} \frac{\ln(|\operatorname{tors}(H_n(G_i\backslash\overline{X}))|)}{[G:G_i]} = 0$ and $\lim_{i\in I} \frac{b_n(G_i\backslash\overline{X};\mathbb{Q})}{[G:G_i]} = 0$, then

$$\lim_{i\in I} \frac{\mathrm{d}\bigl(H_n(G_i\backslash\overline{X})\bigr)}{[G:G_i]} = 0.$$

For the notions of L^2-Betti number $b_n^{(2)}\bigl(\overline{X};\mathcal{N}(G)\bigr)$ and L^2-torsion $\rho^{(2)}\bigl(\overline{X};\mathcal{N}(G)\bigr)$ we refer for instance to [25].

Definition 1.4 (Integral torsion)**.** Define for a finite $\mathbb{Z}$-chain complex D_* its *integral torsion*

$$\rho^{\mathbb{Z}}(D_*) \quad := \quad \sum_{n\ge 0} (-1)^n \cdot \ln\bigl(\bigl|\operatorname{tors}(H_n(D_*))\bigr|\bigr) \quad \in \mathbb{R},$$

where $\bigl|\operatorname{tors}(H_n(D_*))\bigr|$ is the order of the torsion subgroup of the finitely generated abelian group $H_n(D_*)$.

Given a finite CW-complex X, define its *integral torsion* $\rho^{\mathbb{Z}}(X)$ by $\rho^{\mathbb{Z}}(C_*(X))$, where $C_*(X)$ is its cellular $\mathbb{Z}$-complex.

Remark 1.5 (Integral torsion and Milnor's torsion)**.** Let C_* be a finite free $\mathbb{Z}$-chain complex. Fix for each $n \ge 0$ a $\mathbb{Z}$-bases for C_n and for $H_n(C)/\operatorname{tors}(H_n(C))$. They induce $\mathbb{Q}$-basis for $\mathbb{Q}\otimes_{\mathbb{Z}} C_n$ and $H_n\bigl(\mathbb{Q}\otimes_{\mathbb{Z}} C_*\bigr) \cong \mathbb{Q}\otimes_{\mathbb{Z}}\bigl(H_n(C)/\operatorname{tors}(H_n(C))\bigr)$. Then the torsion in the sense of Milnor [27, page 365] is $\rho^{\mathbb{Z}}(C_*)$.

Definition 1.6 (Gottlieb's subgroup of the fundamental group)**.** Let $F = (F, x_0)$ be a pointed CW-complex. Let $G_1(F, x_0) \subseteq \pi_1(F, x_0)$ be the subgroup of elements $[w] \in \pi_1(F, x_0)$ for which there exists a homotopy $h\colon F \times [0,1] \to F$ such that $h(x,0) = h(x,1) = x$ holds for $x \in F$ and the loop at x_0 given by $t \mapsto H(x_0, t)$ represents $[w]$.

1.2. **Connected Lie groups as fibers.** We give an example, where the conditions about $G_1(F)$ are automatically satisfied.

Corollary 1.7 (Connected Lie group as fiber)**.** *Let $L \xrightarrow{j} X \xrightarrow{f} B$ be a fibration with a connected Lie group L as fiber and connected CW-complexes B as basis and X as total space. Consider a homomorphism $\phi\colon \pi_1(X) \to G$ such that the image of $\pi_1(L) \xrightarrow{\pi_1(j)} \pi_1(X) \xrightarrow{\phi} G$ is infinite. Then all assumptions in Theorem 1.1 are satisfied and all the assertions* (1), (2), (3), (4) *and* (5) *hold.*

Proof. Lemma 2.1 (3) implies $G_1(L) = \pi_1(L)$. Because of Theorem 1.1 it remains to show $\rho^{(2)}(\overline{L};\mathcal{N}(H)) = 0$, where H is the image of $\pi_1(L) \xrightarrow{\pi_1(j)} \pi_1(X) \xrightarrow{\phi} G$ and $\overline{L} \to L$ is the covering associated to the induced epimorphism $\pi_1(L) \to H$.

Let $T \subseteq L$ be a maximal torus. Then we have the fibration $T \xrightarrow{i} L \xrightarrow{p} L/T$. Again by Lemma 2.1 (3) we have $G_1(T) = \pi_1(T)$. Hence by Theorem 1.1 (5) it suffices to show $\rho^{(2)}(\overline{T};\mathcal{N}(H)) = 0$, where $\overline{T} \to T$ is the covering associated to the epimorphism $\pi_1(T) \xrightarrow{\pi_1(i)} \pi_1(L) \xrightarrow{\pi_1(j)} \pi_1(X) \xrightarrow{\phi} H$. (This is an epimorphism as $\pi_1(T) \to \pi_1(L)$ is surjective.). Since H is by assumption infinite, we can write H as a product of subgroups $\mathbb{Z} \times A$ and $\overline{T}$ as the product of the $\mathbb{Z}$-space $\widetilde{S^1}$ and of a finite free A-CW-complex Y. Then we conclude from [25, Theorem 3.93 on page 161]

$$\begin{aligned}
\rho^{(2)}(\overline{T};\mathcal{N}(H)) &= \rho^{(2)}\bigl(\widetilde{S^1} \times Y;\mathcal{N}(\mathbb{Z} \times A)\bigr) \\
&= \rho^{(2)}\bigl(\widetilde{S^1};\mathcal{N}(\mathbb{Z})\bigr) \cdot \chi(A\backslash Y) \\
&= 0,
\end{aligned}$$

since $\rho^{(2)}\bigl(\widetilde{S^1};\mathcal{N}(\mathbb{Z})\bigr)$ vanishes by [25, (3.24) on page 136]. □

Example 1.8. Let $L \xrightarrow{j} X \xrightarrow{f} B$ be a fibration with a connected Lie group L as fiber and connected CW-complexes B as basis and X as total space. Suppose that $\pi_2(B)$ vanishes and that $\pi_1(L)$ is infinite. Consider the case $G = \pi_1(X)$, $\phi = \mathrm{id}_G$ and $\overline{X} = \widetilde{X}$. Then all assumptions in Theorem 1.1 are satisfied and all the assertions (1), (2), (3), (4) and (5) hold. This follows from Corollary 1.7 and the long exact homotopy associated to the fibration above.

1.3. **S^1-actions.**

Corollary 1.9 (S^1-action)**.** *Let X be a connected finite S^1-CW-complex. Suppose that for one (and hence all) base points $x \in X$ the composite $\pi_1(S^1) \xrightarrow{\pi_1(\mathrm{ev}_x)} \pi_1(X) \xrightarrow{\phi} G$ has infinite image, where the evaluation map $\mathrm{ev}_x\colon S^1 \to X$ sends z to zx. (This implies that the S^1-action has no fixed points.) Then all the assertions* (1)*,* (2)*,* (3)*,* (4) *and* (5) *appearing in Theorem 1.1 hold.*

Corollary 1.9 follows directly from Corollary 1.7 provided that the action is free and hence we obtain a fibration $S^1 \xrightarrow{\mathrm{ev}_x} X \to S^1\backslash X$ over the finite CW-complex $S^1\backslash X$. We omit the proof of the general case, where one only have to modify Lemma 3.24 a little bit, namely, one uses induction over the skeletons of the S^1-CW-complex X.

Corollary 1.10 (S^1-action on an aspherical closed manifold)**.** *Let M be an aspherical closed manifold with an S^1-action which is non-trivial, i.e., there exists $x \in M$ and $z \in S^1$ with $zx \neq x$. Now take $G = \pi_1(X)$ and $\phi = \mathrm{id}_{\pi_1(X)}$.*

Then all the assertions (1)*,* (2)*,* (3)*,* (4) *and* (5) *appearing in Theorem 1.1 hold for this choice of ϕ.*

Proof. The $\mathrm{ev}_x\colon S^1 \to X$ induces an injection on the fundamental groups [25, Theorem 1.43 on page 48]. Now apply Corollary 1.9. □

1.4. **Approximation Conjectures.** The following two conjectures are motivated by [4, Conjecture 1.3] and [25, Conjecture 11.3 on page 418 and Question 13.52 on page 478]. We will prove them in a special case in Corollary 1.13.

Conjecture 1.11 (Approximation Conjecture for L^2-torsion)**.** *Let X be a finite connected CW-complex and let $\overline{X} \to X$ be a G-covering.*

(1) If the G-CW-structure on $\overline{X}$ and for each $i \in I$ the CW-structure on $G_i\backslash\overline{X}$ come from a given CW-structure on X, then

$$\rho^{(2)}(\overline{X};\mathcal{N}(G)) = \lim_{i\to\infty} \frac{\rho^{(2)}(G_i\backslash\overline{X};\mathcal{N}(\{1\}))}{[G:G_i]};$$

(2) If X is a closed Riemannian manifold and we equip $G_i\backslash\overline{X}$ and $\overline{X}$ with the induced Riemannian metrics, one can replace the torsion in the equality appearing in (1) *by the analytic versions;*

(3) If $b_n^{(2)}(\overline{X};\mathcal{N}(G))$ vanishes for all $n \geq 0$, then

$$\rho^{(2)}(\overline{X};\mathcal{N}(G)) = \lim_{i\to\infty} \frac{\rho^{\mathbb{Z}}(G_i\backslash\overline{X})}{[G:G_i]}.$$

Conjecture 1.12 (Homological growth and L^2-torsion for aspherical closed manifolds)**.** *Let M be an aspherical closed manifold of dimension d and fundamental group $G = \pi_1(M)$. Then*

(1) For any natural number n with $2n \neq d$ we have

$$b_n^{(2)}(\widetilde{M};\mathcal{N}(G)) = \lim_{i\to\infty} \frac{b_n(G_i\backslash\widetilde{M};\mathbb{Q})}{[G:G_i]} = 0.$$

If $d = 2n$ is even, we get

$$b_n^{(2)}(\widetilde{M};\mathcal{N}(G)) = \lim_{i\to\infty} \frac{b_n(G_i\backslash\widetilde{M};\mathbb{Q})}{[G:G_i]} = (-1)^n \cdot \chi(M) \geq 0;$$

(2) For any natural number n with $2n+1 \neq d$ we have

$$\lim_{i\in I} \frac{\ln\big(\big|\operatorname{tors}\big(H_n(G_i\backslash\widetilde{M})\big)\big|\big)}{[G:G_i]} = 0.$$

If $d = 2n+1$, we have

$$\lim_{i\in I} \frac{\ln\big(\big|\operatorname{tors}\big(H_n(G_i\backslash\widetilde{M})\big)\big|\big)}{[G:G_i]} = (-1)^n \cdot \rho^{(2)}\big(\widetilde{M};\mathcal{N}(G)\big) \geq 0.$$

For a brief survey on elementary amenable groups we refer for instance to [25, Section 6.4.1 on page 256ff]. Solvable groups are examples of elementary amenable groups. Every elementary amenable group is amenable, the converse is not true in general.

Some evidence for the two conjectures above comes from

Corollary 1.13. *Let M be an aspherical closed manifold with fundamental group $G = \pi_1(M)$. Suppose that M carries a non-trivial S^1-action or suppose that G contains a non-trivial elementary amenable normal subgroup. Then we get for all $n \geq 0$*

$$\begin{aligned}
\lim_{i\to\infty} \frac{b_n(G_i\backslash\widetilde{M};K)}{[G:G_i]} &= 0;\\
\lim_{i\in I} \frac{\mathrm{d}\big(H_n(G_i\backslash\widetilde{M})\big)}{[G:G_i]} &= 0;\\
\lim_{i\in I} \frac{\ln\big(\big|\operatorname{tors}\big(H_n(G_i\backslash\widetilde{M})\big)\big|\big)}{[G:G_i]} &= 0;\\
\lim_{i\in I} \frac{\rho^{(2)}\big(G_i\backslash\widetilde{M};\mathcal{N}(\{1\})\big)}{[G:G_i]} &= 0;\\
\lim_{i\in I} \frac{\rho^{\mathbb{Z}}\big(G_i\backslash\widetilde{M}\big)}{[G:G_i]} &= 0;\\
b_n^{(2)}(\widetilde{M};\mathcal{N}(G)) &= 0;\\
\rho^{(2)}(\widetilde{M};\mathcal{N}(G)) &= 0.
\end{aligned}$$

In particular Conjecture 1.11 and Conjecture 1.12 are true.

Proof. The case of a non-trivial S^1-action has already been taken care of in Corollary 1.10.

Now suppose that G contains a non-trivial elementary amenable normal subgroup. We conclude from [15, Corollary 2 on page 240] that there is a non-trivial normal abelian subgroup A of G. Since G and hence A are torsionfree, A is infinite. We obtain an exact sequence of groups $1 \to A \to G \to Q$ for $Q = G/A$. Since M is aspherical and hence a model for BG, we obtain a fibration $BA \to M \to BQ$. We conclude $\pi_1(BA) = G_1(BA)$ from Lemma 2.1 (2). Hence we can apply Theorem 1.1 in the case $\phi = \mathrm{id}_G$ and therefore all claims follow except $\rho^{(2)}(\widetilde{M};\mathcal{N}(G)) = 0$ (since we do not know whether BQ is finite.) But $\rho^{(2)}(\widetilde{M};\mathcal{N}(G)) = 0$ has already been proved in [31], see also [19].

□

1.5. **Homological growth of groups.** For a group G we write $H_n(G) := H_n(BG)$ and $b_n(G;K) := b_n(BG;K)$ for the classifying space BG.

Theorem 1.14. *Consider a natural number d and a residually finite group G such that there is a model for BG with finite $(d+1)$-skeleton. Assume either that G contains a normal infinite solvable subgroup or that G is virtually torsionfree with finite virtual cohomological dimension and contains a normal infinite elementary amenable subgroup. Then we get for all $n \leq d$*

$$\begin{aligned}
\lim_{i\in I} \frac{\mathrm{d}(H_n(G_i))}{[G:G_i]} &= 0;\\
\lim_{i\in I} \frac{\ln\bigl(\left|\operatorname{tors}(H_n(G_i))\right|\bigr)}{[G:G_i]} &= 0;\\
\lim_{i\in I} \frac{b_n(G_i;K)}{[G:G_i]} &= 0.
\end{aligned}$$

Proof. Let $S \subseteq G$ be an infinite normal solvable subgroup. The commutator subgroup of a group is a characteristic subgroup. Hence S contains an infinite characteristic subgroup S' whose commutator $[S',S']$ is finite. The subgroup S' is normal in G. Since G is residually finite and $[S',S']$ is finite, we can find a normal subgroup $H' \subseteq G$ of finite index with $H' \cap [S',S'] = \{1\}$. Let $H := \bigcap_{\mu\in\operatorname{aut}(G)} \mu(H')$. Obviously H is a characteristic subgroup of G. Since G is finitely generated and hence contains only finitely many normal subgroup of index $[G:H']$, H has finite index in G. The intersection $A := H \cap S'$ is an infinite normal subgroup of G. Since $H \cap [S',S'] = \{1\}$, the projection $S' \to S'/[S',S']$ is injective on A. Hence A is an infinite abelian normal subgroup of G.

Put $Q = G/A$. We obtain a fibration of connected CW-complexes $BA \to BG \to BQ$. We conclude $G_1(BA) = \pi_1(BA)$ from Lemma 2.1 (2). Now we can apply Theorem 1.1 to the fibration $BA \to BG \to BQ$ for the canonical isomorphism $\phi\colon \pi_1(BG) \to G$.

Let $G' \subseteq G$ be a torsionfree subgroup of finite index whose cohomological dimension is finite. There is an index $i_0 \in I$ such that $G_i \subseteq G'$ holds for $i \geq i_0$. Put $G_i' := G_i \cap G'$. Since $[G:G_i] = [G:G'] \cdot [G':G_i']$ holds for $i \geq i_0$, it suffices to prove the claim for G' and the system $\{G_i' \mid i \in I\}$. The group G' contains a normal infinite abelian subgroup $H' \subseteq G'$ by [15, Corollary 2 on page 240]. Now the claim follows for G' and hence for G from the argument above. □

1.6. **Rank gradient and cost.** Let G be a finitely generated group. Let $H \subseteq G$ be a subgroup of finite index $[G:H]$. Then $\frac{\mathrm{d}(H)-1}{[G:H]} \leq \mathrm{d}(G) - 1$. Therefore the following limit exists and is called *rank gradient* (see Lackenby [16])

$$RG(G;(G_i)_{i\in I}) \quad := \quad \lim_{i\in I} \frac{\mathrm{d}(G_i) - 1}{[G:G_i]}. \tag{1.15}$$

It is not known in general whether this limit is independent of the choice of the system $(G_i)_{i\in I}$.

We have (cf. Lemma 2.13 (3))

$$\begin{aligned}
b_1(BG_i;K) &\leq \mathrm{d}(G_i);\\
\mathrm{d}(H_1(BG_i)) &\leq b_1(BG_i;\mathbb{Q}) + \frac{\ln\bigl(\left|\operatorname{tors}(H_1(BG_i))\right|\bigr)}{\ln(2)}.
\end{aligned}$$

Question 1.16. For which finitely generated groups G, sequences $(G_i)_{i\in I}$ with $\bigcap_{i\in I} G_i = \{1\}$ and fields K, do we have

$$RG(G;(G_i)_{i\in I}) = \limsup_{i\in I} \frac{b_1(G_i;K)}{[G:G_i]}?$$

Question 1.17. For which groups G, does the limit $\lim_{i\in I} \frac{b_1(G_i;K)}{[G:G_i]}$ exist for all systems $(G_i)_{i\in I}$ with $\bigcap_{i\in I} G_i = \{1\}$ and fields K and is independent of the choice of $(G_i)_{i\in I}$ and K?

Abért-Nikolov [2, Theorem 3] have shown for a finitely presented residually finite group G which contains a normal infinite amenable subgroup that $RG(G;(G_i)_{i\in I}) = 0$ holds for all systems $(G_i)_{i\in I}$ (with trivial intersection). Hence the answer to the Questions 1.16 and 1.17 above is yes for such groups.

The questions above is related to questions of Gaboriau (see [11, 12, 13]), whether every essentially free measure preserving Borel action of a group has the same cost, and whether the difference of the cost and the first L^2-Betti number of a measurable equivalence relation is always equal to 1.

The answer to the Questions 1.16 and 1.17 is negative in general if we drop the condition that the system $\{G_i \mid i \in I\}$ has non-trivial intersection, as the following example shows.

Example 1.18. Consider a group H. Put $G = \mathbb{Z} * H$. For a natural number n let $G_n \subseteq G$ be the preimage of $n\cdot\mathbb{Z}$ under the projection $\mathrm{pr}\colon G = \mathbb{Z} * H \to \mathbb{Z}$. We obtain a inverse system of subgroups $\{G_i \mid i = 1,2,3,\ldots\}$ directed by the property i_1 divides i_2 satisfying $\bigcap_{i\geq 1} G_i = \ker(\mathrm{pr})$. Let $BG_i \to BG$ be the covering of BG associated to $G_i \subseteq G$. Then BG_i is homeomorphic to $S^1 \vee \bigvee_{j=1}^i BH$. We have

$$G_i \cong \pi_1(BG_i) \cong \pi_1(S^1 \vee \bigvee_{j=1}^i BH) \cong \mathbb{Z} * *_{j=1}^i H.$$

Since $\mathrm{d}(A*B) = \mathrm{d}(A) + \mathrm{d}(B)$ holds (see [9, Corollary 2 in Section 8.5 on page 227], we conclude

$$\begin{aligned}
H_1(G_i;K) &= K \oplus \bigoplus_{j=1}^i H_1(H;K);\\
H_1(G_i) &= \mathbb{Z} \oplus \bigoplus_{j=1}^i H_1(H);\\
\mathrm{d}(G_i) &= 1 + i\cdot \mathrm{d}(H);\\
\lim_{i\to\infty} \frac{b_1(G_i;K)}{i} &= b_1(H;K);\\
\lim_{i\to\infty} \frac{\mathrm{d}(H_1(G_i))}{i} &= \mathrm{d}(H_1(H));\\
R(G;(G_i)_{i\geq 1}) &= \mathrm{d}(H).
\end{aligned}$$

Let q be a prime different from p. Put $H = \mathbb{Z}/p * \mathbb{Z}/q * \mathbb{Z}/q$. Then $b_1(H;\mathbb{Q}) = 0$, $b_1(H;\mathbb{F}_p) = 1$, $\mathrm{d}(H_1(H)) = 2$, and $\mathrm{d}(H) = 3$. Hence we obtain

$$\lim_{i\to\infty} \frac{b_1(G_i;\mathbb{Q})}{i} < \lim_{i\to\infty} \frac{b_1(G_i;\mathbb{F}_p)}{i} < \lim_{i\to\infty} \frac{\mathrm{d}(H_1(G_i))}{i} < R(G;(G_i)_{i\geq 1}).$$

Obviously BH can be chosen to be of finite type. Let $BH^{(2)}$ be its two-skeleton. Put $X = S^1 \vee BH^{(2)}$. This is a finite 2-dimensional CW-complex. By Maunder's short proof of the Kan-Thurston Theorem (see [26]), we can find a group Γ with a finite 2-dimensional model for $B\Gamma$ together with a map $f\colon B\Gamma \to X$ such that for any local coefficient system M of X the map f induces an isomorphism $f_*\colon H_n(B\Gamma;f^*M) \xrightarrow{\cong} H_n(X;M)$ and the map $\pi_1(f)\colon \pi_1(B\Gamma) = \Gamma \to G = \pi_1(X)$ is surjective. If Z is a connected CW-complex, $H \subseteq \pi_1(Z)$ is a subgroup, $K[\pi_1(Z)/H]$ is the obvious coefficient system on Z, and $\overline{Z} \to Z$ is the H-covering associated to the subgroup $H \subseteq \pi_1(Z)$, then $H_n(\overline{Z};K) \cong H_n(Z;K[\pi_1(Z)/H])$. Let $\Gamma_i \subseteq \Gamma$ be

the preimage of G_i under the epimorphism $\pi_1(f)\colon \pi_1(B\Gamma) = \Gamma \to G := \pi_1(X)$. We conclude

$$\begin{aligned} b_1(\Gamma_i;K) &= b_1(G_i;K); \\ \mathrm{d}(H_1(\Gamma_i)) &= \mathrm{d}(H_1(G_i)); \\ \mathrm{d}(\Gamma_i) &\geq \mathrm{d}(G_i). \end{aligned}$$

Hence we get

$$\lim_{i\to\infty} \frac{b_1(\Gamma_i;\mathbb{Q})}{i} < \lim_{i\to\infty} \frac{b_1(\Gamma_i;\mathbb{F}_p)}{i} < \lim_{i\to\infty} \frac{\mathrm{d}(H_1(G_i))}{i} < R(\Gamma;(\Gamma_i)_{i\geq 1}).$$

The advantage of the more elaborate construction using Maunder's result is that there is a finite 2-dimensional model for $B\Gamma$ and Γ is in particular torsionfree. On the other hand we have no idea what the group Γ is.

Other examples of this kind can be found in [3, 10].

2. Preliminaries

In this section we present some preliminaries for the proof of our main Theorem 1.1.

2.1. Gottlieb's subgroup of the fundamental group. We have defined $G_1(F,x_0) \subseteq \pi_1(F,x_0)$ in Definition 1.6. We collect some basic properties.

If v is a path in F from x_0 to x_1, then the associated isomorphism $c_w\colon \pi_1(F,x) \to \pi_1(F,y)$ given by $[w] \mapsto [v^{-}*w*v]$ induces an isomorphism $c_w'\colon G_1(F,x) \xrightarrow{\cong} G_1(F,y)$ which is the identity in the case $x=y$. Therefore we can and will suppress the base point $x_0 \in F$.

The subgroup $G_1(F)$ was originally defined by Gottlieb [14]. The elementary proof of the next lemma can be found in [14] and [22, Proposition 4.3].

Lemma 2.1. *Let F be a connected CW-complex. Then:*

1. *An element $g \in \pi_1(F)$ belongs to $G_1(F)$ if and only if it belongs to the center of $\pi_1(F)$ and the map $l_g\colon \widetilde{F} \to \widetilde{F}$ on the universal covering given by multiplication with g is $\pi_1(F)$-homotopic to the identity;*
2. *$G_1(F)$ is contained in the center of $\pi_1(F)$. If F is aspherical, $G_1(F)$ agrees with the center of $\pi_1(F)$;*
3. *If F is a connected Lie group, or more generally a connected H-space, then $G_1(F) = \pi_1(F)$;*
4. *If F' is another connected CW-complex, then we obtain a canonical isomorphism*
$$G_1(F\times F') \xrightarrow{\cong} G_1(F)\times G_1(F');$$
5. *If $f\colon F \to F'$ is a pointed homotopy equivalence of connected CW-complexes, then it induces an isomorphism*
$$G_1(F) \xrightarrow{\cong} G_1(F');$$
6. *F is homotopy equivalent to $S^1 \times F'$ for some CW-complex F' if and only there exists an isomorphism $f\colon \mathbb{Z}\times H \to \pi_1(F)$ for some group H such that $f(\mathbb{Z}) \subseteq G_1(F)$.*

2.2. Relating L^2-torsion and $\mathbb{Z}$-torsion. Let C_* be a finite based free $\mathbb{Z}$-chain complex, for instance the cellular chain complex $C_*(X)$ of a finite CW-complex. Let $H_n^{(2)}(C_*^{(2)})$ be the L^2-homology of $C_*^{(2)}$ with respect to the von Neumann algebra $\mathcal{N}(\{1\}) = \mathbb{C}$. The underlying complex vector space is the homology $H_n(\mathbb{C}\otimes_{\mathbb{C}} C_*)$ of $\mathbb{C}\otimes_{\mathbb{Z}} C_*(X)$ but it comes now with the structure of a Hilbert space. For the reader's convenience we recall this Hilbert space structure. Recall that $\mathbb{C}\otimes_{\mathbb{Z}} C_n(X)$ inherits

from the cellular $\mathbb{Z}$-basis on $C_n(X)$ and the standard Hilbert space structure on $\mathbb{C}$ the structure of a Hilbert space and the resulting L^2-chain complex is denoted by $C_*^{(2)}$. Let

$$\Delta_n^{(2)} = \big(c_n^{(2)}\big)^* \circ c_n^{(2)} + c_{n+1}^{(2)} \circ \big(c_{n+1}^{(2)}\big)^* \colon C_n^{(2)} \to C_n^{(2)}$$

be the associated Laplacian. Equip $\ker(\Delta_n^{(2)}) \subseteq C_n^{(2)}$ with the induced Hilbert space structure. Equip $H_n^{(2)}(C_*^{(2)})$ with the Hilbert space structure for which the obvious $\mathbb{C}$-isomorphism $\ker(\Delta_n^{(2)}) \to H_n^{(2)}(C_*^{(2)})$ becomes an isometric isomorphism. This is the same as the Hilbert quotient structure with respect to the projection $\ker\big(c_n^{(2)}\big) \to H_n^{(2)}(C_*^{(2)})$, if we equip $\ker\big(c_n^{(2)}\big) \subseteq C_n^{(2)}$ with the Hilbert subspace structure.

Notation 2.2. If M is a finitely generated abelian group, define

$$M_f \quad := \quad M/\operatorname{tors}(M).$$

Choose a $\mathbb{Z}$-basis on $H_n(C_*)_f$. This and the standard Hilbert space structure on $\mathbb{C}$ induces a Hilbert space structure on $\mathbb{C} \otimes_{\mathbb{Z}} H_n(C_*)_f$. We denote this Hilbert space by $\big(H_n(C_*)_f\big)^{(2)}$. We have the canonical $\mathbb{C}$-isomorphism

$$\alpha_n \colon \big(H_n(C_*)_f\big)^{(2)} \quad \xrightarrow{\cong} \quad H_n^{(2)}(C_*^{(2)}). \tag{2.3}$$

Now we can consider the logarithm of the Fuglede-Kadison determinant

$$\ln\big(\det_{\mathcal{N}(\{1\})}\big(\alpha_n \colon \big(H_n(C_*)_f\big)^{(2)} \to H_n^{(2)}(C_*^{(2)})\big)\big) \in \mathbb{R}.$$

It is independent of the choice of the $\mathbb{Z}$-basis of $H_n(C_*)_f$, since the absolute value of the determinant of an invertible matrix over $\mathbb{Z}$ is always 1.

Lemma 2.4. *Let C_* be a finite based free $\mathbb{Z}$-chain complex. Then*

$$\rho^{\mathbb{Z}}(C_*) - \rho^{(2)}\big(C_*^{(2)}; \mathcal{N}(\{1\})\big) \quad = \quad \sum_{n \geq 0} (-1)^n \cdot \ln\big(\det_{\mathcal{N}(\{1\})}(\alpha_n)\big).$$

Proof. Since $\mathbb{Z}$ is a principal ideal domain and C_* is a finite free $\mathbb{Z}$-chain complex, $H_n(C_*)_f$ is finitely generated free and $\operatorname{tors}(H_n(C_*))$ is a finite product of finite cyclic groups. Denote by $n\big[H_n(C_*)_f\big]$ the $\mathbb{Z}$-chain complex which is concentrated in dimension n and has as nth chain module $H_n(C_*)_f$. In the sequel we equip $H_n(C_*)_f$ with some $\mathbb{Z}$-basis. For every $n \geq 0$, one can find finite based free $\mathbb{Z}$-chain complexes $F_*^{n,i}$ for $i = 1, 2, \ldots, r_n$ such that $F_*^{n,i}$ is concentrated in dimensions $(n+1)$ and n, its $(n+1)$th differential is given by multiplication $\mathbb{Z} \xrightarrow{l_{n,i}} \mathbb{Z}$ with some integer $l_{n,i} \geq 2$ and there is an isomorphism

$$\eta_n \colon H_n(C_*)_f \oplus \bigoplus_{i=1}^{r_n} H_n(F_*^{n,i}) \xrightarrow{\cong} H_n(C_*)$$

of abelian group such that the composite of the inverse of η_n and the canonical projection $H_n(C_*)_f \oplus \bigoplus_{i=1}^{r_n} H_n(F_*^{p,i}) \to H_n(C_*)_f$ agrees with the canonical projection $H_n(C_*) \to H_n(C_*)_f$. Using the exact sequence of $\mathbb{Z}$-modules $C_{n+1} \xrightarrow{c_{n+1}} \ker(c_n) \to H_n(C_*) \to 0$ one constructs a $\mathbb{Z}$-chain map

$$f_* \colon \bigoplus_{n \geq 0} \left(n\big[H_n(C_*)_f\big] \oplus \bigoplus_{i=1}^{r_n} F_*^{n,i} \right) \to C_*$$

such that $H_n(f_*)$ is the isomorphism η_n. Since the source and the target of f_* are free $\mathbb{Z}$-chain complexes, f_* is a $\mathbb{Z}$-chain homotopy equivalence. The L^2-torsion

$t^{(2)}\big(f_*^{(2)}\big)$ of $f_*^{(2)}$ is defined in [25, Definition 3.31 on page 141]. We conclude from [25, Theorem 3.35 (5) on page 142])

$$t^{(2)}\big(f_*^{(2)}\big) = \rho^{(2)}\big(C_*^{(2)}\big) - \rho^{(2)}\left(\bigoplus_{n\geq 0}\left(n\big[H_n(C_*)_f\big] \oplus \bigoplus_{i=1}^{r_n} F_*^{n,i}\right)\right) + \sum_{n\geq 0}(-1)^n \cdot \ln\Big(\det\big(H_n(f_*^{(2)})\big)\Big). \tag{2.5}$$

Since f_* is a $\mathbb{Z}$-chain homotopy equivalence and hence its Whitehead torsion is an element in $\{\pm 1\}$, we get

$$t^{(2)}\big(f_*^{(2)}\big) = \ln(|\pm 1|) = 0. \tag{2.6}$$

We obtain an equality of maps of finitely generated Hilbert $\mathcal{N}(\{1\})$-modules

$$H_n\big(f_*^{(2)}\big) \quad = \quad \alpha_n. \tag{2.7}$$

One easily checks

$$\rho^{(2)}\left(\bigoplus_{n\geq 0}\left(n\big[H_n(C_*)_f\big]^{(2)} \oplus \bigoplus_{i=1}^{r_n}(F_*^{n,i})^{(2)}\right)\right) = \sum_{n\geq 0}\sum_{i=1}^{r_n}\rho^{(2)}\Big(\big(F_*^{n,i}\big)^{(2)};\mathcal{N}(\{1\})\Big); \tag{2.8}$$

$$\rho^{\mathbb{Z}}(C_*) \quad = \quad \sum_{n\geq 0}\sum_{i=1}^{r_n}\rho^{\mathbb{Z}}\big(F_*^{n,i}\big). \tag{2.9}$$

Because of (2.5), (2.6), (2.7), (2.8) and (2.9) it suffices to show

$$\rho^{(2)}\Big(\big(F_*^{n,i}\big)^{(2)}\Big) \quad = \quad \rho^{\mathbb{Z}}\big(F_*^{n,i}\big).$$

This is done by the following calculation.

$$\begin{aligned}
\rho^{(2)}\big((F_*^{n,i})^{(2)}\big) &= (-1)^n \cdot \ln\Big(\det_{\mathcal{N}(\{1\})}\big(l_{n,i}\colon \mathbb{C} \to \mathbb{C}\big)\Big)\\
&= (-1)^n \cdot \ln(l_{i,n})\\
&= (-1)^n \cdot \ln\Big(\big|\mathrm{tors}\big(H_n(F_*^{n,i})\big)\big|\Big)\\
&= \rho^{\mathbb{Z}}\big(F_*^{n,i}\big).
\end{aligned}$$

This finishes the proof of Lemma 2.4. □

Notation 2.10. Let A be a finitely generated free abelian group and let $B \subseteq A$ be a subgroup. Define the *closure* of B in A to be the subgroup

$$\overline{B} \quad = \quad \{x \in A \mid n \cdot x \in B \text{ for some non-zero integer } n\}.$$

Lemma 2.11. *Let $u\colon \mathbb{Z}^r \to \mathbb{Z}^s$ be a homomorphism of abelian groups. Let $j_k\colon \ker(u) \to \mathbb{Z}^r$ be the inclusion and $\mathrm{pr}_c\colon \mathbb{Z}^s \to \mathrm{coker}(u)_f$ be the canonical projection. Choose $\mathbb{Z}$-basis for $\ker(u)$ and $\mathrm{coker}(u)_f$.*

Then $\det_{\mathcal{N}(\{1\})}\big(j_k^{(2)}\big)$ and $\det_{\mathcal{N}(\{1\})}\big(\mathrm{pr}_c^{(2)}\big)$ are independent of the choice of the $\mathbb{Z}$-basis for $\ker(u)$ and $\mathrm{coker}(u)_f$, and we have

$$\det_{\mathcal{N}(\{1\})}(u^{(2)}) \quad = \quad \det_{\mathcal{N}(\{1\})}\big(j_k^{(2)}\big) \cdot \big|\mathrm{tors}(\mathrm{coker}(u))\big| \cdot \det_{\mathcal{N}(\{1\})}\big(\mathrm{pr}_c^{(2)}\big),$$

and

$$\begin{array}{rcccl}
1 & \leq & \det_{\mathcal{N}(\{1\})}(j_k^{(2)}) & \leq & \det_{\mathcal{N}(\{1\})}(u^{(2)});\\
1 & \leq & \det_{\mathcal{N}(\{1\})}\big(\mathrm{pr}_c^{(2)}\big) & \leq & \det_{\mathcal{N}(\{1\})}(u^{(2)});\\
1 & \leq & \big|\mathrm{tors}(\mathrm{coker}(u))\big| & \leq & \det_{\mathcal{N}(\{1\})}(u^{(2)}).
\end{array}$$

Proof. Because of [25, Lemma 13.12 on page 459] the trivial group satisfies the Determinant Conjecture. This implies $1 \le \det_{\mathcal{N}(\{1\})}(u^{(2)})$ and $1 \le \det_{\mathcal{N}(\{1\})}\bigl(\mathrm{pr}_c^{(2)}\bigr)$ and that for any isomorphism $j\colon \mathbb{Z}^l \to \mathbb{Z}^l$ we have $\det_{\mathcal{N}(\{1\})}(j^{(2)}) = 1$. The latter together with [25, Theorem 3.14 (1) on page 128] implies that the $\mathbb{Z}$-basis for $\ker(u)$ and $\operatorname{coker}(u)_f$ do not matter and that it suffices to show the equation

$$\det_{\mathcal{N}(\{1\})}(u^{(2)}) \quad = \quad \det_{\mathcal{N}(\{1\})}\bigl(j_k^{(2)}\bigr) \cdot \bigl|\mathrm{tors}(\operatorname{coker}(u))\bigr| \cdot \det_{\mathcal{N}(\{1\})}\bigl(\mathrm{pr}_c^{(2)}\bigr).$$

Let $\overline{u}$ be the map of finitely generated free abelian groups $\mathbb{Z}^r/\ker(f) \to \overline{\operatorname{im}(u)}$ induced by u. Equip $\mathbb{Z}^r/\ker(f)$ and $\overline{\operatorname{im}(u)}$ with $\mathbb{Z}$-basis. Let $\mathrm{pr}_k\colon \mathbb{Z}^r \to \mathbb{Z}^r/\ker(u)$ be the canonical projection and $j_i\colon \overline{\operatorname{im}(u)} \to \mathbb{Z}^s$ be the inclusion. We conclude from [25, Theorem 3.14 (1) on page 128]

$$\det_{\mathcal{N}(\{1\})}(u^{(2)}) = \det_{\mathcal{N}(\{1\})}\bigl(j_i^{(2)}\bigr) \cdot \det_{\mathcal{N}(\{1\})}\bigl(\overline{u}^{(2)}\bigr) \cdot \det_{\mathcal{N}(\{1\})}\bigl(\mathrm{pr}_k^{(2)}\bigr).$$

Hence it remains to show

$$\begin{aligned} \det_{\mathcal{N}(\{1\})}\bigl(j_i^{(2)}\bigr) &= \det_{\mathcal{N}(\{1\})}\bigl(\mathrm{pr}_c^{(2)}\bigr); \\ \det_{\mathcal{N}(\{1\})}\bigl(\mathrm{pr}_k^{(2)}\bigr) &= \det_{\mathcal{N}(\{1\})}\bigl(j_k^{(2)}\bigr). \end{aligned}$$

This follows from Lemma 2.4 applied to the 2-dimensional acyclic based free finite $\mathbb{Z}$-chain complexes $\ker(u) \xrightarrow{j_k} \mathbb{Z}^r \xrightarrow{\mathrm{pr}_k} \mathbb{Z}^r/\ker(u)$ and $\overline{\operatorname{im}(u)} \xrightarrow{j_i} \mathbb{Z}^1 \xrightarrow{\mathrm{pr}_c} \operatorname{coker}(u)_f$. This finishes the proof of Lemma 2.11. □

2.3. **Minimal numbers of generators.**

Definition 2.12 (Minimal numbers of generators)**.** Let G be a finitely generated group. Denote by $\mathrm{d}(G) \in \mathbb{N}$ the *minimal numbers of generators* of G, where we put $\mathrm{d}(\{1\}) = 0$.

If M is an abelian group, then $\mathrm{d}(M)$ is the minimum over all natural numbers n for which there exists an epimorphism of abelian groups $\mathbb{Z}^n \to M$.

Lemma 2.13.
1. *Let $r \ge 0$, $s \ge 0$ and $d_1 \ge 2, d_2 \ge 2, \dots, d_s \ge 2$ be integers with $d_1|d_2|d_3|\cdots|d_s$ and $M \cong \mathbb{Z}^r \oplus \bigoplus_{i=1}^s \mathbb{Z}/d_i$. Then*
$$\mathrm{d}(M) = r + s;$$
2. *Let M be a finitely generated abelian group. For any prime p we can write the p-Sylow subgroup $\mathrm{tors}(M)_p$ of the finite abelian group $\mathrm{tors}(M)$ as a direct sum*
$$\mathrm{tors}(M)_p \cong \bigoplus_{i=1}^{s_p} \mathbb{Z}/p^{k_i}$$
for integers $k_i \ge 1$ and $s_p \ge 0$. Then
$$\mathrm{d}(M) = \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} M) + \max\{s_p \mid p \text{ prime}\}.$$
In particular
$$\mathrm{d}(M) = \mathrm{d}(M/\mathrm{tors}(M)) + \mathrm{d}(\mathrm{tors}(M));$$
3. *If K is any field, we have*
$$\dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} M) \le \dim_K(K \otimes_{\mathbb{Z}} M) \le \mathrm{d}(M) \le \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} M) + \frac{\ln\bigl(|\mathrm{tors}(M)|\bigr)}{\ln(2)};$$
4. *If $0 \to M_0 \to M_1 \to M_2 \to 0$ is an exact sequence of finitely generated $\mathbb{Z}$-modules, then*
$$\mathrm{d}(M_1) \le \mathrm{d}(M_0) + \mathrm{d}(M_2).$$

If M is a submodule of the finitely generated $\mathbb{Z}$-module M, then

$$\mathrm{d}(M) \le \mathrm{d}(N);$$

If N is a quotient module of the finitely generated $\mathbb{Z}$-module M, then

$$\mathrm{d}(N) \le \mathrm{d}(M).$$

Proof. (1) Consider any epimorphism $f\colon \mathbb{Z}^n \to M$. We have to show $n \ge r+s$. The kernel of f is a finitely generated free $\mathbb{Z}$-module of rank $m \le n$. Hence we obtain an exact sequence $0 \to \mathbb{Z}^m \xrightarrow{j} \mathbb{Z}^n \xrightarrow{f} M \to 0$. By composing i with appropriate $\mathbb{Z}$-isomorphisms from the left and the right we can arrange that there are integers $c_1 \ge 1$, $c_2 \ge 1, \dots, c_m \ge 1$ such that $c_1 | c_2 | c_3 | \dots | c_m$, and j sends the element e_i, whose entries are all zero except the ith entry which is 1, to $c_i \cdot e_i$ for $i = 1, 2 \dots, m$. Let $t \in \{0, 1, 2, \dots, m\}$ be the integer for which $c_i = 1$ for $i \le t$. Then

$$M \cong \mathbb{Z}^{n-m} \oplus \bigoplus_{i=t+1}^{m} \mathbb{Z}/c_i.$$

From the structure theorem for abelian groups we conclude $n-m = r$ and $m-t = s$. This implies

$$r + s = n - m + m - t \le n.$$

(2) and (3) These follow from assertion (1).

(4) Let $0 \to M_0 \xrightarrow{j} M_1 \xrightarrow{q} M_2 \to 0$ be an exact sequence of finitely generated $\mathbb{Z}$-modules. Choose epimorphisms $f_i\colon \mathbb{Z}^{\mathrm{d}(M_i)} \to M_i$ for $i = 0, 2$. Choose a homomorphism $\widetilde{f_2}\colon \mathbb{Z}^{\mathrm{d}(M_2)} \to M_2$ with $q \circ \widetilde{f_2} = f_2$. Define the $\mathbb{Z}$-homomorphism $f_1\colon \mathbb{Z}^{\mathrm{d}(M_0)+\mathrm{d}(M_2)} = \mathbb{Z}^{\mathrm{d}(M_0)} \oplus \mathbb{Z}^{\mathrm{d}(M_2)} \to M_1$ by $(j \circ f_0) \oplus \widetilde{f_2}$. One easily checks that f_1 is surjective. Hence

$$\mathrm{d}(M_1) \le \mathrm{d}(M_0) + \mathrm{d}(M_2).$$

Let M be a submodule of the finitely generated $\mathbb{Z}$-module N. Choose an epimorphism $f\colon \mathbb{Z}^{\mathrm{d}(N)} \to N$. Then f induces an epimorphism $f^{-1}(M) \to M$ and $f^{-1}(M)$ is isomorphic to $\mathbb{Z}^m$ for some integer $m \le \mathrm{d}(N)$. Hence $\mathrm{d}(M) \le m \le \mathrm{d}(N)$. The claim for quotient modules is obvious. □

Remark 2.14. Notice that the minimal number of generators is not additive under exact sequences, actually it is not even additive under direct sums. Namely, if p and q are different prime numbers, then $\mathrm{d}(\mathbb{Z}/p \oplus \mathbb{Z}/q) = \mathrm{d}(\mathbb{Z}/p) = \mathrm{d}(\mathbb{Z}/q) = 1$. Moreover, for non-abelian groups it is not true that $\mathrm{d}(H) \le \mathrm{d}(G)$ holds for a subgroup $H \subseteq G$, e.g., this fails for G the free group of two generators and H any proper subgroup of finite index.

Remark 2.15. The main reason why Betti numbers and the number of minimal generators are easier to handle than the order of the torsion in homology is that the minimal number of generators satisfies assertion (4) of Lemma 2.13 and the dimension of K-vector spaces satisfies the analogous statement. This is not true for the order of the torsion in a finitely generated abelian group. Passing to the quotient of a finitely generated abelian group may increase the order of the torsion submodule as the example $\mathbb{Z} \to \mathbb{Z}/n$ shows.

Lemma 2.16. *Let G be a finite group. Let M be a $\mathbb{Z}G$-module which is finitely generated as abelian group. Consider $n \ge 1$. Let d_n be an integer such that there is a free $\mathbb{Z}G$-resolution F_* of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ satisfying $\dim_{\mathbb{Z}G}(F_n) \le d_n$. ($F_*$ may depend on n.)*

Then $H_n(G; M)$ is annihilated by multiplication with $|G|$ and we get

$$\begin{array}{rcl} \mathrm{d}(H_n(G;M)) & \le & d_n \cdot \mathrm{d}(M); \\ |H_n(G;M)| & \le & |G|^{d_n \cdot \mathrm{d}(M)}. \end{array}$$

Proof. Fix $n \geq 1$. Then $H_n(G;M)$ is annihilated by multiplication with $|G|$ by [5, Corollary 10.2 on page 84]. By definition $H_n(G;M) = H_n(M \otimes_{\mathbb{Z}G} F_*)$. We conclude from Lemma 2.13 (4)

$$\begin{aligned} \mathrm{d}\big(H_n(G;M)\big) = \mathrm{d}\big(H_n(M \otimes_{\mathbb{Z}G} F_*)\big) \leq \mathrm{d}\big(M \otimes_{\mathbb{Z}G} F_n\big) \\ = \mathrm{d}\Big(M^{\dim_{\mathbb{Z}G}(F_n)}\Big) \leq \dim_{\mathbb{Z}G}(F_n) \cdot \mathrm{d}(M) \leq d_n \cdot \mathrm{d}(M). \end{aligned}$$

Consider an epimorphism $f \colon \mathbb{Z}^l \to H_n(G;M)$. Since multiplication with $|G|$ annihilates $H_n(G;M)$, it induces an epimorphism $\overline{f} \colon (\mathbb{Z}/|G|)^l \to H_n(G;M)$. This implies $|H_n(G;M)| \leq |G|^l$. We conclude

$$|H_n(G;M)| \leq |G|^{\mathrm{d}(H_n(G;M))} \leq |G|^{d_n \cdot \mathrm{d}(M)}.$$

□

2.4. **Nilpotent modules.**

Definition 2.17 (Nilpotent $\mathbb{Z}G$-module)**.** We call a $\mathbb{Z}G$-module M *nilpotent*, if there exists an integer $r \geq 0$ and a filtration $\{0\} = M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots \subseteq M_r = M$ such that the G-action on M_i/M_{i-1} is trivial for $i = 1, 2, \ldots, r$. The minimal number r for which such a filtration exists is called the *filtration length* of M.

A $\mathbb{Z}G$-module is trivial if and only if it is nilpotent of filtration length 0. The G-action on an $\mathbb{Z}G$-module M is trivial if and only if it is nilpotent of filtration length at most 1. The elementary proof of the next lemma is left to the reader.

Lemma 2.18. *Let G be a group. Then:*

1. *A $\mathbb{Z}G$-submodule of a nilpotent $\mathbb{Z}G$-module of filtration length r is again nilpotent and has filtration length $\leq r$;*
2. *A $\mathbb{Z}G$-quotient module of a nilpotent $\mathbb{Z}G$-module of filtration length r is again nilpotent and has filtration length $\leq r$;*
3. *Let $0 \to M_0 \to M_1 \to M_2 \to 0$ be an exact sequence of $\mathbb{Z}G$-modules. If two of them are nilpotent, then all three are nilpotent and the filtration length of M_1 is less or equal to the sum of the filtration length of M_0 and M_2;*
4. *If $\{M_i \mid i \in I\}$ is a family of nilpotent $\mathbb{Z}G$-modules of filtration length $\leq r$ for some index set I, then $\bigoplus_{i \in I} M_i$ is a nilpotent $\mathbb{Z}G$-module of filtration length $\leq r$.*

3. Proof of the main Theorem 1.1

In this section we present the proof of Theorem 1.1.

The following Subsections 3.1, 3.2 and 3.3 deal with the problem to give estimates of our invariants of interest for a $\mathbb{Z}G$-chain complex C_* with nilpotent homology in terms of their values for $\mathbb{Z} \otimes_{\mathbb{Z}G} C_*$, where G is a finite abelian group. The main result will be Proposition 3.13.

3.1. **Passage from $\mathbb{Z} \otimes_{\mathbb{Z}G} H_n(C_*)$ to $H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} C_*)$.**

Lemma 3.1. *Let G be a finite group. Let C_* be a finitely generated free $\mathbb{Z}G$-chain complex. Let $\nu_n = \nu_n(C_*) \colon \mathbb{Z} \otimes_{\mathbb{Z}G} H_n(C_*) \to H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} C_*)$ be the canonical map.*

Then

$$\begin{aligned}
\bigl|\ker(\nu_n)\bigr| &\le \prod_{p=1}^{n} \bigl|H_{p+1}(G;H_{n-p}(C_*))\bigr|;\\
\bigl|\operatorname{coker}(\nu_n)\bigr| &\le \prod_{p=1}^{n} \bigl|H_{p}(G;H_{n-p}(C_*))\bigr|;\\
\bigl|\mathrm{d}\bigl(\mathbb{Z}\otimes_{\mathbb{Z}G} H_n(C_*)\bigr) - \mathrm{d}\bigl(H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*)\bigr)\bigr| &\le \sum_{p=1}^{n} \mathrm{d}\bigl(H_p(G;H_{n-p}(C_*))\bigr)\\
&\quad + \sum_{p=1}^{n} \mathrm{d}\bigl(H_{p+1}(G;H_{n-p}(C_*))\bigr).
\end{aligned}$$

Proof. The map $\nu_n\colon \mathbb{Z}\otimes_{\mathbb{Z}G} H_n(C_*) = \operatorname{Tor}_0^{\mathbb{Z}G}(H_n(C_*),\mathbb{Z}) \to H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*)$ is the edge homomorphism in the universal coefficient spectral sequence whose E^2-term is $\operatorname{Tor}_p^{\mathbb{Z}G}(H_q(C_*),\mathbb{Z}) = H_p(G;H_q(C_*))$ and which converges to $H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*)$. From this spectral sequence we obtain an exact sequence of abelian groups

$$0 \to A \to \mathbb{Z}\otimes_{\mathbb{Z}G} H_n(C_*) \xrightarrow{\nu_n} H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*) \to B \to 0,$$

where the modules A and B have filtrations $\{0\} \subseteq A_1 \subseteq \cdots \subseteq A_n = A$ and $\{0\} \subseteq B_1 \subseteq \cdots \subseteq B_n = B$ such that A_n is a sub-quotient of $H_{p+1}(G;H_{n-p}(C_*))$ and B_n is a sub-quotient of $H_p(G;H_{n-p}(C_*))$ for each $p \in \{1,2,\dots,n\}$. In particular we get from Lemma 2.13 (4)

$$\begin{aligned}
|A| &\le \prod_{p=1}^{n} \bigl|H_{p+1}(G;H_{n-p}(C_*))\bigr|;\\
|B| &\le \prod_{p=1}^{n} \bigl|H_{p}(G;H_{n-p}(C_*))\bigr|;\\
\mathrm{d}(A) &\le \sum_{p=1}^{n} \mathrm{d}\bigl(H_{p+1}(G;H_{n-p}(C_*))\bigr);\\
\mathrm{d}(B) &\le \sum_{p=1}^{n} \mathrm{d}\bigl(H_{p}(G;H_{n-p}(C_*))\bigr);\\
\mathrm{d}(A) &\ge \mathrm{d}\bigl(\mathbb{Z}\otimes_{\mathbb{Z}G} H_n(C_*)\bigr) - \mathrm{d}\bigl(H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*)\bigr);\\
\mathrm{d}(B) &\ge \mathrm{d}\bigl(H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*)\bigr) - \mathrm{d}\bigl(\mathbb{Z}\otimes_{\mathbb{Z}G} H_n(C_*)\bigr).
\end{aligned}$$

Now Lemma 3.1 follows. □

3.2. **Passage from M to $\mathbb{Z}\otimes_{\mathbb{Z}G} M$.**

Lemma 3.2. *Let G be a finite group. Let M be a nilpotent $\mathbb{Z}G$-module. Let $r \ge 1$ be an integer such that the filtration length of M is less or equal to r. Then the kernel of the canonical epimorphism $\mu = \mu(M)\colon M \to \mathbb{Z}\otimes_{\mathbb{Z}G} M$ is finite and we have*

$$\begin{aligned}
\bigl|\ker(\mu(M))\bigr| &\le |G|^{(r-1)\cdot \mathrm{d}(G)\cdot \mathrm{d}(M)};\\
\mathrm{d}(M) &\le r\cdot(\mathrm{d}(G)+1)^{r-1}\cdot \mathrm{d}(\mathbb{Z}\otimes_{\mathbb{Z}G} M).
\end{aligned}$$

Proof. We use induction over the filtration length r of M. The induction beginning $r = 1$ is obviously true. The induction step from $r-1$ to $r \ge 2$ is done as follows.

Choose an exact sequence of $\mathbb{Z}G$-modules $0 \to M' \to M \to M'' \to 0$ such that M' is nilpotent of filtration length $\le r-1$ and G acts trivially on M''. We have

the following commutative diagram with exact rows

$$\begin{tikzcd}
 & 0 \arrow[r] & M' \arrow[r] \arrow[d, "\mu(M')"] & M \arrow[r] \arrow[d, "\mu(M)"] & M'' \arrow[r] \arrow[d, "\mu(M'')", "\cong"'] & 0 \\
 & H_1(G;M'') \arrow[r] & \mathbb{Z} \otimes_{\mathbb{Z}G} M' \arrow[r] & \mathbb{Z} \otimes_{\mathbb{Z}G} M \arrow[r] & \mathbb{Z} \otimes_{\mathbb{Z}G} M'' \arrow[r] & 0
\end{tikzcd}$$

It induces the exact sequence

$$0 \to \ker(\mu(M')) \to \ker(\mu(M)) \to \operatorname{im}\bigl(H_1(G;M'') \to \mathbb{Z} \otimes_{\mathbb{Z}G} M'\bigr) \to 0. \tag{3.3}$$

Let $\{s_1, s_2, \ldots, s_{\mathrm{d}(G)}\}$ be a finite set of generators of G with minimal numbers of generators. We obtain an exact sequence of $\mathbb{Z}G$-modules

$$\mathbb{Z}G^{\mathrm{d}(G)} \xrightarrow{(s_1-1, s_2-1, \ldots, s_{\mathrm{d}(G)}-1)} \mathbb{Z}G \to \mathbb{Z} \to 0,$$

which can be extended to the left to a free $\mathbb{Z}G$-resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. Because of Lemma 2.13 (4) this implies

$$\begin{aligned}\mathrm{d}\bigl(H_1(G,M'')\bigr) \le \mathrm{d}\bigl(\mathbb{Z}G^{\mathrm{d}(G)} \otimes_{\mathbb{Z}G} M''\bigr) &\le \mathrm{d}(G) \cdot \mathrm{d}(M'') \\ &= \mathrm{d}(G) \cdot \mathrm{d}(\mathbb{Z} \otimes_{\mathbb{Z}G} M'') \le \mathrm{d}(G) \cdot \mathrm{d}(\mathbb{Z} \otimes_{\mathbb{Z}G} M).\end{aligned} \tag{3.4}$$

As multiplication with G annihilates $H_1(G,M)$ we get

$$|H_1(G,M'')| \quad \le \quad |G|^{\mathrm{d}(\mathbb{Z} \otimes_{\mathbb{Z}G} M) \cdot \mathrm{d}(G)}. \tag{3.5}$$

This implies

$$|H_1(G,M'')| \quad \le \quad |G|^{\mathrm{d}(M) \cdot \mathrm{d}(G)}. \tag{3.6}$$

By induction hypothesis

$$\bigl|\ker(\mu(M'))\bigr| \quad \le \quad |G|^{(r-2) \cdot \mathrm{d}(G) \cdot \mathrm{d}(M')}. \tag{3.7}$$

We conclude from (3.3), (3.6) and (3.7) using Lemma 2.13 (4)

$$\begin{aligned}\bigl|\ker(\mu(M))\bigr| &\le \bigl|\ker(\mu(M'))\bigr| \cdot |H_1(G;M'')| \\ &\le |G|^{(r-2) \cdot \mathrm{d}(G) \cdot \mathrm{d}(M')} \cdot |G|^{\mathrm{d}(M'') \cdot \mathrm{d}(G)} \\ &\le |G|^{(r-2) \cdot \mathrm{d}(G) \cdot \mathrm{d}(M)} \cdot |G|^{\mathrm{d}(M) \cdot \mathrm{d}(G)} \\ &= |G|^{(r-1) \cdot \mathrm{d}(G) \cdot \mathrm{d}(M)}.\end{aligned}$$

Next we estimate using Lemma 2.13 (4), (3.3) and (3.4)

$$\begin{aligned}\mathrm{d}(M) &\le \mathrm{d}\bigl(\ker(\mu(M))\bigr) + \mathrm{d}(\mathbb{Z} \otimes_{\mathbb{Z}G} M) \\ &\le \mathrm{d}\bigl(\ker(\mu(M'))\bigr) + \mathrm{d}(H_1(G;M'')) + \mathrm{d}(\mathbb{Z} \otimes_{\mathbb{Z}G} M) \\ &\le \mathrm{d}\bigl(\ker(\mu(M'))\bigr) + \mathrm{d}(G) \cdot \mathrm{d}(\mathbb{Z} \otimes_{\mathbb{Z}G} M) + \mathrm{d}(\mathbb{Z} \otimes_{\mathbb{Z}G} M) \\ &\le \mathrm{d}(M') + (\mathrm{d}(G)+1) \cdot \mathrm{d}(\mathbb{Z} \otimes_{\mathbb{Z}G} M).\end{aligned} \tag{3.8}$$

We estimate using (3.4) and Lemma 2.13 (4)

$$\begin{aligned}\mathrm{d}(\mathbb{Z} \otimes_{\mathbb{Z}G} M') &\le \mathrm{d}(H_1(G;M'')) + \mathrm{d}(\mathbb{Z} \otimes_{\mathbb{Z}G} M) \\ &\le \mathrm{d}(G) \cdot \mathrm{d}(\mathbb{Z} \otimes_{\mathbb{Z}G} M) + \mathrm{d}(\mathbb{Z} \otimes_{\mathbb{Z}G} M) \\ &= (\mathrm{d}(G)+1) \cdot \mathrm{d}(\mathbb{Z} \otimes_{\mathbb{Z}G} M).\end{aligned} \tag{3.9}$$

By induction hypothesis

$$\mathrm{d}(M') \quad \le \quad (r-1) \cdot (m(G)+1)^{r-2} \cdot \mathrm{d}(\mathbb{Z} \otimes_{\mathbb{Z}G} M'). \tag{3.10}$$

We conclude from (3.8), (3.9) and (3.10)

$$\begin{aligned}
\mathrm{d}(M) &\le \mathrm{d}(M') + (\mathrm{d}(G)+1)\cdot \mathrm{d}(\mathbb{Z}\otimes_{\mathbb{Z}G} M)\\
&\le (r-1)\cdot (m(G)+1)^{r-2}\cdot \mathrm{d}(\mathbb{Z}\otimes_{\mathbb{Z}G} M') + (\mathrm{d}(G)+1)\cdot \mathrm{d}(\mathbb{Z}\otimes_{\mathbb{Z}G} M)\\
&\le (r-1)\cdot (m(G)+1)^{r-2}\cdot (\mathrm{d}(G)+1)\cdot \mathrm{d}(\mathbb{Z}\otimes_{\mathbb{Z}G} M)\\
&\qquad\qquad\qquad + (\mathrm{d}(G)+1)\cdot \mathrm{d}(\mathbb{Z}\otimes_{\mathbb{Z}G} M)\\
&= \bigl((r-1)\cdot (m(G)+1)^{r-2}+1\bigr)\cdot (\mathrm{d}(G)+1)\cdot \mathrm{d}(\mathbb{Z}\otimes_{\mathbb{Z}G} M)\\
&\le r\cdot (m(G)+1)^{r-2}\cdot (\mathrm{d}(G)+1)\cdot \mathrm{d}(\mathbb{Z}\otimes_{\mathbb{Z}G} M)\\
&= r\cdot (m(G)+1)^{r-1}\cdot \mathrm{d}(\mathbb{Z}\otimes_{\mathbb{Z}G} M).
\end{aligned}$$

This finishes the proof of Lemma 3.2. □

Lemma 3.11. *Let G be a finite abelian group. Then there exists a free $\mathbb{Z}G$-resolution F_* of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$ satisfying for $n \ge 0$*

$$\dim_{\mathbb{Z}G}(F_n) = \binom{n+\mathrm{d}(G)-1}{\mathrm{d}(G)-1}.$$

Proof. Put $m = \mathrm{d}(G)$. Then we get decomposition

$$G = \bigoplus_{i=1}^{m} \mathbb{Z}/d_i$$

as direct sum of finite cyclic groups from Lemma 2.13 (1). There is a free $\mathbb{Z}[\mathbb{Z}/d_i]$-resolution $F_*^{(i)}$ of the trivial $\mathbb{Z}[\mathbb{Z}/d_i]$-module $\mathbb{Z}$ such that $\dim_{\mathbb{Z}[\mathbb{Z}/d_i]}(F_n^{(i)}) = 1$ for all $n \ge 0$. Then $F_* = \bigotimes_{i=1}^m F_*^{(i)}$ is a free $\mathbb{Z}G$-resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. Obviously $\dim_{\mathbb{Z}G}(F_n)$ is the number of weak compositions of n into m non-negative integers, i.e., the number of k-tuples $(i_1, i_2, \dots, i_m)$ of non-negative integers such that $i_1 + i_2 + \dots + i_m = n$. Hence

$$\dim_{\mathbb{Z}G}(F_n) = \binom{n+m-1}{m-1}.$$

□

Remark 3.12. Lemma 3.11 is the reason why we often assume the existence of an infinite normal subgroup $A \subseteq G$ which is abelian. The estimate we get and will need is polynomial in $\mathrm{d}(G)$. There is an obvious polynomial estimate in terms of $|G|$ coming from the bar resolution which works for all finite groups, but it is not sufficient for us.

3.3. Relating $H_n(C_*)$ to $H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*)$.

Proposition 3.13. *There exists functions*

$$C_0, C_1, D_0, D_1 \colon \bigl\{(r,n,p) \in \mathbb{N}^3 \mid p \le n\bigr\} \quad\to\quad \mathbb{R},$$

such that the following is true:

Let G be a finite abelian group. Let C_* be a finitely generated free $\mathbb{Z}G$-chain complex. Suppose that $H_p(C_*)$ is a nilpotent $\mathbb{Z}G$-module of filtration length $\le r$ for $p = 0, 1, 2, \dots, n$. Then:

(1) We have

$$\mathrm{d}(H_n(C_*)) \le \sum_{p=0}^{n} C_0(r,n,p)\cdot \mathrm{d}(G)^{C_1(r,n,p)}\, \mathrm{d}\bigl(H_p(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*)\bigr);$$

(2) We have

$$\begin{aligned}
\ln\left(\left|\ker(H_n(\mathrm{pr}_*))\right|\right) &\le \sum_{p=0}^{n} D_0(r,n,p)\cdot \ln(|G|)\cdot \mathrm{d}(G)^{D_1(r,n,p)}\cdot \mathrm{d}\big(H_p(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*)\big);\\
\ln\left(\left|\operatorname{coker}(H_n(\mathrm{pr}_*))\right|\right) &\le \sum_{p=0}^{n} D_0(r,n,p)\cdot \ln(|G|)\cdot \mathrm{d}(G)^{D_1(r,n,p)}\cdot \mathrm{d}\big(H_p(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*)\big),
\end{aligned}$$

where $\mathrm{pr}_*\colon C_* \to \mathbb{Z}\otimes_{\mathbb{Z}G} C_*$ *is the canonical projection.*

Proof. (1) Fix a natural number r. We define $C(r,n,p)$ for $p\in\{0,1,2,\dots,n\}$ by induction over n. In the sequel we abbreviate $m=\mathrm{d}(G)$.

Firstly we explain the induction beginning $n=0$. Then $\nu_0\colon \mathbb{Z}\otimes_{\mathbb{Z}G} H_0(C_*)\to H_0(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*)$ is an isomorphism since we always assume $C_n=0$ for $p\le -1$. We conclude from Lemma 3.2

$$\begin{aligned}
\mathrm{d}(H_0(C_*)) &\le r\cdot(m+1)^{r-1}\cdot \mathrm{d}\big(\mathbb{Z}\otimes_{\mathbb{Z}G} H_0(C_*)\big)\\
&= r\cdot(m+1)^{r-1}\cdot \mathrm{d}\big(H_0(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*)\big)\\
&\le r\cdot 2^{r-1}\cdot m^{r-1}\cdot \mathrm{d}\big(H_0(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*)\big).
\end{aligned}$$

So we put $C_0(r,0,0)=r\cdot 2^{r-1}$ and $C_1(r,0,0)=r-1$.

The induction beginning from $(n-1)$ for $n\ge 1$ is done as follows. We conclude from Lemma 3.2

$$\begin{aligned}
\mathrm{d}(H_n(C_*)) &\le r\cdot(m+1)^{r-1}\cdot \mathrm{d}\big(\mathbb{Z}\otimes_{\mathbb{Z}G} H_n(C_*)\big)\\
&\le r\cdot 2^{r-1}\cdot m^{r-1}\cdot \mathrm{d}\big(\mathbb{Z}\otimes_{\mathbb{Z}G} H_n(C_*)\big).
\end{aligned}\tag{3.14}$$

We estimate using Lemma 2.16, Lemma 3.1, and Lemma 3.11

$$\begin{aligned}
&\mathrm{d}\big(\mathbb{Z}\otimes_{\mathbb{Z}G} H_n(C_*)\big)\\
&\le \mathrm{d}\big(H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*)\big)+\sum_{p=1}^{n}\mathrm{d}\big(H_p(G;H_{n-p}(C_*))\big)\\
&\qquad +\sum_{p=1}^{n}\mathrm{d}\big(H_{p+1}(G;H_{n-p}(C_*))\big)\\
&\le \mathrm{d}\big(H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*)\big)+\sum_{p=1}^{n}\binom{p+m-1}{m-1}\cdot \mathrm{d}\big(H_{n-p}(C_*))\big)\\
&\qquad +\sum_{p=1}^{n}\binom{p+m}{m-1}\cdot \mathrm{d}\big(H_{n-p}(C_*))\big)\\
&= \mathrm{d}\big(H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*)\big)\\
&\qquad +\sum_{p=0}^{n-1}\left(\binom{n-p+m-1}{m-1}+\binom{n-p+m}{m-1}\right)\cdot \mathrm{d}\big(H_p(C_*))\big).
\end{aligned}\tag{3.15}$$

Putting (3.14) and (3.15) together yields

$$\begin{aligned}
&\mathrm{d}\big(H_n(C_*)\big)\le r\cdot 2^{r-1}\cdot m^{r-1}\cdot \mathrm{d}\big(H_n(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*)\big)\\
&+\sum_{p=0}^{n-1} r\cdot 2^{r-1}\cdot m^{r-1}\cdot\left(\binom{n-p+m-1}{m-1}+\binom{n-p+m}{m-1}\right)\cdot \mathrm{d}\big(H_p(C_*))\big).
\end{aligned}\tag{3.16}$$

We estimate for $p = 0, 1, 2 \ldots, n$

$$\binom{n-p+m}{m-1} = \frac{(n-p+m)!}{(m-1)! \cdot (n-p+1)!} \leq (n-p+m) \cdot (n-p+m-1) \cdot \cdots \cdot m \leq (n-p+m)^{n-p+1} \leq (n+m)^{n+1} \leq n^{n+1} \cdot m^{n+1}. \tag{3.17}$$

Analogously we get for $p = 0, 1, 2 \ldots, n$

$$\binom{n-p+m-1}{m-1} \leq n^{n+1} \cdot m^{n+1}. \tag{3.18}$$

We conclude from (3.16), (3.17) and (3.18)

$$\mathrm{d}\big(H_n(C_*)\big) \leq r \cdot 2^{r-1} \cdot m^{r-1} \cdot \mathrm{d}\big(H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} C_*)\big) + \sum_{p=0}^{n-1} r \cdot 2^{r-1} \cdot m^{r-1} \cdot 2 \cdot n^{n+1} \cdot m^{n+1} \cdot \mathrm{d}\big(H_p(C_*)\big)\big). \tag{3.19}$$

By induction hypothesis we get for $p = 0, 1, 2, \ldots, (n-1)$

$$\mathrm{d}(H_p(C_*)) \leq \sum_{i=0}^{p} C_0(r,p,i) \cdot m^{C_1(r,p,i)} \cdot \mathrm{d}\big(H_i(\mathbb{Z} \otimes_{\mathbb{Z}G} C_*)\big). \tag{3.20}$$

Putting (3.19) and (3.20) together yields

$$\begin{aligned}
\mathrm{d}\big(H_n(C_*)\big) &\leq r \cdot 2^{r-1} \cdot m^{r-1} \cdot \mathrm{d}\big(H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} C_*)\big) \\
&\quad + \sum_{p=0}^{n-1} r \cdot 2^{r-1} \cdot m^{r-1} \cdot 2 \cdot n^{n+1} \cdot m^{n+1} \cdot \\
&\qquad \left(\sum_{i=0}^{p} C_0(r,p,i) \cdot m^{C_1(r,p,i)} \cdot \mathrm{d}\big(H_i(\mathbb{Z} \otimes_{\mathbb{Z}G} C_*)\big)\right) \\
&\leq r \cdot 2^{r-1} \cdot m^{r-1} \cdot \mathrm{d}\big(H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} C_*)\big) \\
&\quad + \sum_{p=0}^{n-1} \left(\sum_{i=p}^{n-1} r \cdot 2^{r} \cdot n^{n+1} \cdot C_0(r,i,p) \cdot m^{n+r+C_1(r,i,p)}\right) \\
&\qquad \cdot \mathrm{d}\big(H_p(\mathbb{Z} \otimes_{\mathbb{Z}G} C_*)\big) \\
&\leq r \cdot 2^{r-1} \cdot m^{r-1} \cdot \mathrm{d}\big(H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} C_*)\big) \\
&\quad + \sum_{p=0}^{n-1} \left(\sum_{i=p}^{n-1} r \cdot 2^{r} \cdot n^{n+1} \cdot C_0(r,i,p)\right) \\
&\qquad \cdot m^{n+r+\max\{C_1(r,i,p) \mid p \leq i \leq n-1)\}} \cdot \mathrm{d}\big(H_p(\mathbb{Z} \otimes_{\mathbb{Z}G} C_*)\big).
\end{aligned}$$

Define

$$C_0(r,n,p) := \begin{cases} \sum_{i=p}^{n-1} r \cdot 2^r \cdot n^{n+1} \cdot C_0(r,i,p); & \text{if } 0 \leq p \leq n-1; \\ r \cdot 2^{r-1} & \text{if } p = n; \end{cases}$$

$$C_1(r,n,p) := \begin{cases} n + r + \max\{C_1(r,i,p) \mid p \leq i \leq n-1)\}; & \text{if } 0 \leq p \leq n-1; \\ r-1 & \text{if } p = n. \end{cases}$$

Then we get

$$\mathrm{d}(H_n(C_*)) \le \sum_{p=0}^{n} C_0(r,n,p) \cdot m^{C_1(r,n,p)} \cdot \mathrm{d}\big(H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} C_*)\big).$$

This finishes the proof of assertion (1).

(2) We estimate using Lemma 2.16, Lemma 3.1, Lemma 3.11, and (3.18)

$$\begin{array}{rcl} |\operatorname{coker}(\nu_n(C_*))| & \le & \prod_{p=1}^{n} \big|H_p(G; H_{n-p}(C_*))\big| \\ & \le & \prod_{p=1}^{n} |G|^{\binom{p+m-1}{m-1} \cdot \mathrm{d}(H_{n-p}(C_*))} \\ & = & |G|^{\sum_{p=1}^{n} (\binom{p+m-1}{m-1} \cdot \mathrm{d}(H_{n-p}(C_*))} \\ & \le & |G|^{\sum_{p=1}^{n} n^{n+1} \cdot m^{n+1} \cdot \mathrm{d}(H_{n-p}(C_*))}. \end{array}$$

This implies together with assertion (1)

(3.21)

$$\begin{array}{rl} \multicolumn{2}{l}{\ln\big(|\operatorname{coker}(\nu_n(C_*))|\big)} \\ \le & \ln(|G|) \cdot \sum_{p=1}^{n} n^{n+1} \cdot m^{n+1} \cdot \mathrm{d}(H_{n-p}(C_*)) \\ \le & \ln(|G|) \cdot \sum_{p=1}^{n} n^{n+1} \cdot m^{n+1} \cdot \left(\sum_{i=0}^{n-p} C_0(r,n-p,i) \cdot m^{C_1(r,n-p,i)}\, \mathrm{d}\big(H_i(\mathbb{Z} \otimes_{\mathbb{Z}G} C_*)\big) \right) \\ = & \ln(|G|) \cdot \sum_{p=0}^{n-1} \left(\sum_{i=1}^{n-p} n^{n+1} \cdot m^{n+1} \cdot C_0(r,n-i,p) \cdot m^{C_1(r,n-i,p)} \right) \cdot \mathrm{d}\big(H_p(\mathbb{Z} \otimes_{\mathbb{Z}G} C_*)\big) \\ = & \sum_{p=0}^{n-1} \left(\sum_{i=1}^{n-p} n^{n+1} \cdot C_0(r,n-i,p) \cdot \ln(|G|) \cdot m^{n+1+C_1(r,n-i,p)} \right) \cdot \mathrm{d}\big(H_p(\mathbb{Z} \otimes_{\mathbb{Z}G} C_*)\big). \end{array}$$

Analogously we get

(3.22)

$$\begin{array}{rl} \multicolumn{2}{l}{\ln\big(|\ker(\nu_n(C_*))|\big)} \\ \le & \sum_{p=0}^{n-1} \left(\sum_{i=1}^{n-p} n^{n+1} \cdot C_0(r,n-i,p) \cdot \ln(|G|) \cdot m^{n+1+C_1(r,n-i,p)} \right) \cdot \mathrm{d}\big(H_p(\mathbb{Z} \otimes_{\mathbb{Z}G} C_*)\big). \end{array}$$

We estimate using Lemma 3.2 and assertion (1).

(3.23)

$$\begin{array}{rl} \multicolumn{2}{l}{\ln\big(\big|\ker(\mu(H_n(C_*))\big|\big)} \\ \le & \ln(|G|) \cdot (r-1) \cdot m \cdot \mathrm{d}(H_n(C_*)) \\ \le & \ln(|G|) \cdot (r-1) \cdot m \cdot \sum_{p=0}^{n} C_0(r,n,p) \cdot m^{C_1(m,n,p)} \cdot \mathrm{d}\big(H_p(\mathbb{Z} \otimes_{\mathbb{Z}G} C_*)\big) \\ \le & \sum_{p=0}^{n} (r-1) \cdot C_0(r,n,p) \cdot \ln(|G|) \cdot m^{1+C_1(m,n,p)} \cdot \mathrm{d}\big(H_p(\mathbb{Z} \otimes_{\mathbb{Z}G} C_*)\big). \end{array}$$

Since the canonical map $H_n(\mathrm{pr}_*)\colon H_n(C_*) \to H_n(\mathbb{Z} \otimes_{\mathbb{Z}G} C_*)$ agrees with the composition $\nu_n(C_*) \circ \mu(H_n(C_*))$ and $\mu_n(H_n(C_*))$ is surjective, we obtain an exact

sequence

$$0 \to \ker\bigl(\mu(H_n(C_*))\bigr) \to \ker(H_n(\mathrm{pr}_*)) \to \ker\bigl(\nu(C_*))\bigr) \to 0,$$

and an isomorphism

$$\operatorname{coker}(H_n(\mathrm{pr}_*)) \quad \cong \quad \operatorname{coker}\bigl(\nu(C_*))\bigr).$$

We conclude

$$\begin{aligned}
\ln\left(\bigl|\ker(H_n(\mathrm{pr}_*))\bigr|\right) &\leq \ln\left(\bigl|\ker\bigl(\mu(H_n(C_*))\bigr)\bigr|\right) + \ln\left(\bigl|\ker\bigl(\nu_n(C_*))\bigr)\bigr|\right);\\
\ln\left(\bigl|\operatorname{coker}(H_n(\mathrm{pr}_*))\bigr|\right) &= \ln\left(\bigl|\operatorname{coker}\bigl(\nu_n(C_*))\bigr)\bigr|\right).
\end{aligned}$$

This together with (3.21), (3.22), and (3.23) implies

$$\begin{aligned}
&\ln\left(\bigl|\ker(H_n(\mathrm{pr}_*))\bigr|\right)\\
&\leq \sum_{p=0}^{n}(r-1)\cdot C_0(r,n,p)\cdot \ln(|G|)\cdot m^{1+C_1(m,n,p)}\cdot \mathrm{d}\bigl(H_p(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*)\bigr)\\
&\qquad + \sum_{p=0}^{n-1}\left(\sum_{i=1}^{n-p} n^{n+1}\cdot C_0(r,n-i,p)\cdot m^{n+1+C_1(r,n-i,p)}\right)\cdot \ln(|G|)\\
&\qquad\qquad \cdot \mathrm{d}\bigl(H_p(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*)\\
&\leq \sum_{p=0}^{n}(r-1)\cdot C_0(r,n,p)\cdot \ln(|G|)\cdot m^{1+C_1(m,n,p)}\cdot \mathrm{d}\bigl(H_p(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*)\bigr)\\
&\qquad + \sum_{p=0}^{n-1}\left(\sum_{i=1}^{n-p} n^{n+1}\cdot C_0(r,n-i,p)\right)\cdot \ln(|G|)\\
&\qquad\qquad \cdot m^{n+1+\max\{C_1(r,n-i,p)\mid i=1,\ldots n-p\}}\cdot \mathrm{d}\bigl(H_p(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*)\bigr),
\end{aligned}$$

and

$$\begin{aligned}
&\ln\bigl(|\operatorname{coker}(H_n(\mathrm{pr}_*))|\bigr)\\
&\leq \sum_{p=0}^{n-1}\left(\sum_{i=1}^{n-p} n^{n+1}\cdot C_0(r,n-i,p)\right)\cdot \ln(|G|)\\
&\qquad \cdot m^{n+1+\max\{C_1(r,n-i,p)\mid i=1,\ldots n-p\}}\cdot \mathrm{d}\bigl(H_p(\mathbb{Z}\otimes_{\mathbb{Z}G} C_*)\bigr).
\end{aligned}$$

Now assertion (2) follows for the obvious choices of $D_0(r,n,p)$ and $D_1(r,n,p)$. This finishes the proof of Proposition 3.13. $\square$

3.4. **Nilpotent homology.** In this subsection we explain how the condition that our CW-complex under consideration fibers in a specific way enters the proof. Essentially the condition about Gottlieb's subgroup of the fundamental group ensures that later we get on each level nilpotent homology groups what will be needed in order to apply Proposition 3.13.

Lemma 3.24. *Let $F \xrightarrow{j} X \xrightarrow{f} B$ be a fibration, where F and B are connected CW-complexes. Let d be the dimension of B. Consider an epimorphism $\phi\colon \pi_1(X) \to G$. Let $A \subseteq G$ be the image of $G_1(F)$ under the composite $\phi\circ\pi_1(j)\colon \pi_1(F) \to G$. Let $p\colon \overline{X} \to X$ be the G-covering associated to ϕ.*

Then A is a normal abelian subgroup of G and the induced A-action on $\overline{X}$ turns $H_n(\overline{X})$ into a $\mathbb{Z}A$-module which is nilpotent of filtration length $\leq d+1$.

Proof. We use induction over d. The induction beginning $d=-1$ is trivial since then $\overline{X}$ is empty. The induction step from $(d-1)$ to $d \geq 0$ is done as follows. Since B is connected, it is homotopy equivalent to a finite CW-complex with precisely

one 0-cell. Hence we can assume without loss of generality that each skeleton B_d is connected. Choose a pushout

$$\begin{tikzcd} \coprod_{i\in I} S^{d-1} \arrow[r, "\coprod_{i\in I} q_i"] \arrow[d] & B_{d-1} \arrow[d] \\ \coprod_{i\in I} D^d \arrow[r, "\coprod_{i\in I} Q_i"] & B \end{tikzcd}$$

Since f is a fibration and p is a G-covering, we obtain by the pullback construction a G-pushout with a cofibration as left vertical arrow (see [23, Lemma 1.26])

$$\begin{tikzcd} \coprod_{i\in I} q_i^*\overline{X} \arrow[r, "\coprod_{i\in I} \overline{q_i}"] \arrow[d] & (f\circ p)^{-1}(B_{d-1}) \arrow[d] \\ \coprod_{i\in I} Q_i^*\overline{X} \arrow[r, "\coprod_{i\in I} \overline{Q_i}"] & \overline{X} \end{tikzcd}$$

Let $w \in \pi_1(X)$. Then the pointed fiber transport along w yields a pointed homotopy equivalence $\sigma(w)\colon F \to F$ which is unique up to pointed homotopy equivalence (see [21, Section 6]). The induced map $\pi_1(\sigma(w))$ sends $G_1(F)$ to $G_1(F)$ by Lemma 2.1 (5). The following diagram commutes

$$\begin{tikzcd} \pi(F) \arrow[r, "\pi_1(\sigma(w))"] \arrow[d, "\pi_1(j)"] & \pi_1(F) \arrow[d, "\pi_1(j)"] \\ \pi_1(X) \arrow[r, "c_w"] & \pi_1(X) \end{tikzcd}$$

where c_w is conjugation with w. Since $\phi\colon \pi_1(X) \to G$ is by assumption surjective, this implies that $A \subseteq G$ is normal.

Let H be the image of $\phi\circ\pi_1(j)\colon \pi_1(F) \to G$. Obviously $A \subseteq H$. Let $\overline{F} \to F$ be the covering associated to the epimorphism $\pi_1(F) \to H$ induced by $\phi\circ\pi_1(j)$. Since D^d is contractible, we obtain for each $i \in I$ a G-homotopy equivalence of pairs

$$\left(Q_i^*\overline{X}, q_i^*\overline{X}\right) \xrightarrow{\simeq} G \times_H \overline{F} \times (D^d, S^{d-1}).$$

Hence we obtain a long exact sequence of $\mathbb{Z}G$-modules

$$\cdots \to \bigoplus_{i\in I} \mathbb{Z}G \otimes_{\mathbb{Z}H} H_k(\overline{F}) \to H_{d+k}\big((f\circ p)^{-1}(B_{d-1})\big) \to H_{d+k}(\overline{X}) \\ \to \bigoplus_{i\in I} \mathbb{Z}G \otimes_{\mathbb{Z}H} H_k(\overline{F}) \to \cdots .$$

Now we view this as an exact sequence of $\mathbb{Z}A$-modules. By induction hypothesis $H_n\big((f\circ p)^{-1}(B_{d-1})\big)$ is nilpotent of filtration length $\le d$ for all n. The A-operation on $H_l(\overline{F})$ is trivial for all l, since for each $w \in \pi_1(F)$ the $\pi_1(F)$-map $l_w\colon \widetilde{F} \to \widetilde{F}$ given by multiplication with w is $\pi_1(F)$-equivariantly homotopic to the identity because of Lemma 2.1 (1). Hence the left A-action on $\mathbb{Z}G \otimes_{\mathbb{Z}H} H_n(\overline{F})$ is trivial since A is normal in G. This implies that $\mathbb{Z}G \otimes_{\mathbb{Z}H} H_n(\overline{X})$ is a nilpotent $\mathbb{Z}A$-module of filtration length ≤ 1. We conclude from Lemma 2.18 that $H_n(\overline{X})$ is a nilpotent $\mathbb{Z}A$-module of filtration length $\le (d+1)$ for all n. $\square$

3.5. **A priori bounds.** If the sequences under consideration converge, they have to be bounded what we prove next.

Notation 3.25. Let C_* be a finite based free chain complex with differential c_*. Denote by $(C[i]_*, c[i]_*)$ the finite based free $\mathbb{Z}$-chain complex $\mathbb{Z} \otimes_{\mathbb{Z}G_i} C_* = \mathbb{Z}[G/G_i] \otimes_{\mathbb{Z}G} C_*$.

Lemma 3.26. *Consider a finite based free $\mathbb{Z}G$-chain complex C_*. Then there is a constant $\Lambda > 0$ satisfying:*

(1) *For all $i \in I$ and $n \geq 0$ we have*
$$0 \leq \frac{\mathrm{d}(H_n(C[i]_*))}{[G:G_i]} \leq \Lambda;$$
(2) *For all $i \in I$ and $n \geq 0$ we have*
$$0 \leq \frac{\ln\bigl(\det_{\mathcal{N}(\{1\})}\bigl(c[i]_n^{(2)}\bigr)\bigr)}{[G:G_i]} \leq \Lambda;$$
(3) *For all $i \in I$ and $n \geq 0$ we have*
$$\mathrm{d}\bigl(H_n(C[i]_*)\bigr) \leq \Lambda,$$
and
$$0 \leq \frac{\ln\bigl(\bigl|\mathrm{tors}(H_n(C[i]_*))\bigr|\bigr)}{[G:G_i]} \leq \Lambda;$$
(4) *For all $i \in I$ and $n \geq 0$ we have*
$$-\Lambda \leq \frac{\ln\bigl(\det_{\mathcal{N}(\{1\})}(\alpha[i]_n)\bigr)}{[G:G_i]} \leq \Lambda.$$
where $\alpha[i]_n$ is the map associated to $C[i]_*$ in (2.3).

Proof. We conclude from [25, Lemma 13.33 on page 466] that for every natural number n there exists a constant $K_n \geq 0$ such that

(3.27) $$||c[i]_n^{(2)}|| \quad \leq \quad K_n$$

holds for each $i \in I$. Put
$$\begin{aligned}\Lambda_0 \quad &:= \quad \sum_{n=0}^{\dim(C_*)} \max\{\ln(K_n), 1\} \cdot \dim_{\mathbb{Z}G}(C_n);\\ \Lambda \quad &= \quad 4 \cdot \Lambda_0.\end{aligned}$$

Now we can prove the various assertions appearing in Lemma 3.26.

(1) We conclude from Lemma 2.13 (4)
$$\begin{aligned}\frac{\mathrm{d}(H_n(C_*))}{[G:G_i]} \quad &\leq \quad \frac{\dim_{\mathbb{Z}}\bigl(\dim_{\mathbb{Z}}(C[i]_n)\bigr)}{[G:G_i]}\\ &= \quad \dim_{\mathbb{Z}G}(C_n)\\ &\leq \quad \Lambda_0.\end{aligned}$$

(2) We conclude from from (3.27)
$$\begin{aligned}\det_{\mathcal{N}(\{1\})}\bigl(c[i]_n^{(2)}\bigr) \quad &\leq \quad ||c[i]_n^{(2)}||^{\dim_{\mathcal{N}(\{1\}}(C[i]_n^{(2)})}\\ &\leq \quad (K_n)^{\dim_{\mathbb{Z}}(C[i]_n)}.\end{aligned}$$

We have $1 \leq \det_{\mathcal{N}(\{1\}}\bigl(c[i]_n^{(2)}\bigr)$ since the trivial groups satisfy the Determinant Conjecture because of [25, Lemma 13.12 on page 459]. Hence we get
$$1 \leq \det_{\mathcal{N}(\{1\})}\bigl(c[i]_n^{(2)}\bigr) \leq (K_n)^{\dim_{\mathbb{Z}}(C[i]_n)}.$$

Since $\dim_{\mathbb{Z}G}(C_n) = \frac{\dim_{\mathbb{Z}}(C[i]_n)}{[G:G_i]}$, this implies for all $i \in I$ and $n \geq 0$

(3.28) $$0 \leq \frac{\ln(\det_{\mathcal{N}(\{1\})}(c[i]_n^{(2)}))}{[G:G_i]} \leq \Lambda_0 \leq \Lambda.$$

(3) We conclude from Lemma 2.11 and (3.28)

$$\begin{aligned}
\text{(3.29)}\qquad \frac{\ln\left(\left|\operatorname{tors}(H_n(C[i]_*))\right|\right)}{[G:G_i]} &= \frac{\ln\left(\left|\operatorname{tors}(\operatorname{coker}(c[i]_{n+1}))\right|\right)}{[G:G_i]} \\
&\le \frac{\left|\ln\left(\det_{\mathcal{N}(\{1\})}\left(c[i]^{(2)}_{n+1}\right)\right)\right|}{[G:G_i]} \\
&\le \Lambda_0.
\end{aligned}$$

This implies for all $i \in I$ and $n \ge 0$

$$0 \le \frac{\ln\left(\left|\operatorname{tors}(H_n(C[i]_*))\right|\right)}{[G:G_i]} \le \Lambda.$$

(4) Let $\Delta_n \colon C_n \to C_n$ be the combinatorial Laplacian of the finite based free $\mathbb{Z}G$-chain complex C_*. Then $\Delta[i]_n \colon C[i]_n \to C[i]_n$ is the combinatorial Laplacian of the finite based free $\mathbb{Z}$-chain complex $C[i]_*$, and $\Delta[i]^{(2)}_n \colon C[i]^{(2)}_n \to C[i]^{(2)}_n$ is the same as the Laplacian on the $\mathcal{N}(\{1\})$-Hilbert chain complex $C[i]^{(2)}_*$. Let $j[i]\colon \ker(\Delta[i]_n) \to C[i]_n$ be the inclusion. It induces a map $j[i]^{(2)}\colon \ker(\Delta[i]_n)^{(2)} \to C[i]^{(2)}_n$ whose image is contained in $\ker\left(\Delta[i]^{(2)}_n\right)$.

Obviously we have $\ker(c[i]_n) \cap \ker\left(c[i]^*_{n+1}\right) \subseteq \ker(\Delta[i]_n)$. If $k \cdot x \in \ker(c[i]_n) \cap \ker\left(c[i]^*_{n+1}\right)$ for $k \in \mathbb{Z}, k \neq 0$ and $x \in \ker(\Delta[i]_n)$, then $x \in \ker(c[i]_n) \cap \ker\left(c[i]^*_{n+1}\right)$ since $C[i]_*$ is free as $\mathbb{Z}$-module. Since $\ker\left(\Delta[i]^{(2)}_n\right) = \ker\left(c[i]^{(2)}_n\right) \cap \ker\left((c[i]^{(2)}_{n+1})^*\right)$ (see [25, Lemma 1.18 on page 24]), the finitely generated free abelian groups $\ker(c[i]_n) \cap \ker\left(c[i]^*_{n+1}\right)$ and $\ker(\Delta[i]_n)$ have the same rank. This implies

$$\text{(3.30)}\qquad \ker(\Delta[i]_n) = \ker(c[i]_n) \cap \ker\left(c[i]^*_{n+1}\right).$$

Let $u[i]\colon \ker(\Delta[i]_n) \to H_n(C[i]_*)_f$ be the composite of the inclusion $\ker(\Delta[i]_n) \to \ker(c[i]_n)$ with the obvious projection $\ker(c[i]_n) \to H_n(C[i]_*)_f$. Let $v[i]\colon \ker\left(\Delta[i]^{(2)}_n\right) \to H^{(2)}_n\left(C[i]^{(2)}_*\right)$ be the composite of the inclusion $\ker\left(\Delta[i]^{(2)}_n\right) \to \ker\left(c[i]^{(2)}_n\right)$ and the projection $\ker\left(c[i]^{(2)}_n\right) \to H^{(2)}_n\left(C[i]^{(2)}_*\right)$. The following diagram of isomorphisms of $\mathcal{N}(\{1\})$-Hilbert modules commutes

$$\begin{tikzcd}
\ker(\Delta[i]_n)^{(2)} \arrow[r, "{j[i]^{(2)}}"] \arrow[d, "{u[i]^{(2)}}"] & \ker\left(\Delta[i]^{(2)}_n\right) \arrow[d, "{v[i]}"] \\
\left(H_n(C[i]_*)_f\right)^{(2)} \arrow[r, "{\alpha[i]_n}"] & H^{(2)}_n\left(C[i]^{(2)}_*\right)
\end{tikzcd}$$

We conclude from [25, Theorem 3.14 (1) on page 128]

$$\text{(3.31)}\qquad \det_{\mathcal{N}(\{1\})}(\alpha[i]_n) \cdot \det_{\mathcal{N}(\{1\})}(u[i]^{(2)}) = \det_{\mathcal{N}(\{1\})}(v[i]) \cdot \det_{\mathcal{N}(\{1\})}(j[i]^{(2)}).$$

Since $v[i]$ is by definition an isometric isomorphism, we have

$$\text{(3.32)}\qquad \det_{\mathcal{N}(\{1\})}(v[i]) = 1.$$

We conclude from Lemma 2.11

$$\text{(3.33)}\qquad 1 \le \det_{\mathcal{N}(\{1\})}(j[i]^{(2)}) \le \det_{\mathcal{N}(\{1\})}\left(\Delta[i]^{(2)}_n\right);$$

$$\text{(3.34)}\qquad \det_{\mathcal{N}(\{1\})}(u[i]^{(2)}) = |\operatorname{coker}(u[i])|.$$

The following diagram commutes

$$\begin{array}{ccccc}
\ker(c[i]_n) \cap \ker\bigl(c[i]^*_{n+1}\bigr) & \xrightarrow{k[i]_1} & \ker(c[i]_n) & \xrightarrow{q[i]_1} & \ker(c[i]_n)/\overline{\operatorname{im}(c[i]_{n+1})} \\
\downarrow{\scriptstyle l[i]_1} & & \downarrow{\scriptstyle l[i]_2} & & \downarrow{\scriptstyle l[i]_3} \\
\ker\bigl(c[i]^*_{n+1}\bigr) & \xrightarrow{k[i]_2} & C[i]_n & \xrightarrow{q[i]_2} & C[i]_n/\overline{\operatorname{im}(c[i]_{n+1})}
\end{array}$$

where the maps $k[i]_m$ and $l[i]_m$ are inclusions and the maps $q[i]_m$ are projections and $\overline{\operatorname{im}(c[i]_{n+1})}$ has been introduced in Notation 2.10. An easy diagram chase using $\overline{\operatorname{im}(c[i]_{n+1})} \subseteq \ker(c[i]_n)$ shows that $l[i]_3$ induces a map $\overline{l[i]_3}\colon \operatorname{coker}\bigl(q[i]_1 \circ k[i]_1\bigr) \to \operatorname{coker}\bigl(q[i]_2 \circ k[i]_2\bigr)$ such that $\overline{l_3}$ is injective. This implies

$$|\operatorname{coker}(q[i]_1 \circ k[i]_1))| \quad \le \quad |\operatorname{coker}(q[i]_2 \circ k[i]_2)|\,. \tag{3.35}$$

The map $u[i]$ agrees with $q[i]_1 \circ k[i]_1$. We conclude from (3.34) and (3.35)

$$\begin{aligned}\det_{\mathcal{N}(\{1\})}(u[i]^{(2)}) &= |\operatorname{coker}(u[i])| \\ &= |\operatorname{coker}(q[i]_1 \circ k[i]_1)| \\ &\le \bigl|\operatorname{coker}(q[i]_2 \circ k[i]_2)\bigr|\,.\end{aligned} \tag{3.36}$$

Let $\iota[i]\colon \ker(c[i]^*_{n+1}) \oplus \overline{\operatorname{im}(c[i]_{n+1})} \to C[i]_n$ be the inclusion, $k[i]_3\colon \overline{\operatorname{im}(c[i]_{n+1})} \to C[i]_n$ be the inclusion, and $q[i]_3\colon C[i]_n \to C[i]_n/\ker(c[i]^*_{n+1})$ be the projection. The maps $\iota[i]$ and $q[i]_3 \circ k[i]_3$ are injective and have finite cokernels. We conclude from Lemma 2.11

$$\begin{aligned}\bigl|\operatorname{coker}(q[i]_2 \circ k[i]_2\bigr| &= \bigl|\operatorname{coker}(\iota[i])\bigr| \\ &= \bigl|\operatorname{coker}(q[i]_3 \circ k[i]_3)\bigr| \\ &= \det_{\mathcal{N}(\{1\})}\Bigl(\bigl(q[i]_3 \circ k[i]_3\bigr)^{(2)}\Bigr)\,.\end{aligned} \tag{3.37}$$

The map $c[i]^*_{n+1}\colon C[i]_n \to C[i]_{n+1}$ induces an injection $\overline{c[i]^*_{n+1}}\colon C[i]_n/\ker(c[i]^*_{n+1}) \to C[i]_{n+1}$. Let $\overline{c[i]_{n+1}}\colon C[i]_{n+1} \to \overline{\operatorname{im}(c[i]_{n+1})}$ be the map induced by $c[i]_{n+1}$. Then the composite

$$C[i]_{n+1} \xrightarrow{\overline{c[i]_{n+1}}} \overline{\operatorname{im}(c[i]_{n+1})} \xrightarrow{k[i]_3} C[i]_n \xrightarrow{q[i]_3} C[i]_n/\ker(c[i]^*_{n+1}) \xrightarrow{\overline{c[i]^*_{n+1}}} C[i]_{n+1}$$

agrees with the composite $c[i]^*_{n+1} \circ c[i]_{n+1}$. We conclude from [25, Theorem 3.14 (1) on page 128 and Lemma 3.15 (4) on page 129]

$$\begin{aligned}&\Bigl(\det_{\mathcal{N}(\{1\})}\bigl(c[i]^{(2)}_{n+1}\bigr)\Bigr)^2 \\
&= \det_{\mathcal{N}(\{1\})}\bigl((c[i]^{(2)}_{n+1})^* \circ c[i]^{(2)}_{n+1}\bigr) \\
&= \det_{\mathcal{N}(\{1\})}\Bigl(\bigl(c[i]^*_{n+1} \circ c[i]_{n+1}\bigr)^{(2)}\Bigr) \\
&= \det_{\mathcal{N}(\{1\})}\Bigl(\bigl(\overline{c[i]^*_{n+1}} \circ (q[i]_3 \circ k[i]_3) \circ \overline{c[i]_{n+1}}\bigr)^{(2)}\Bigr) \\
&= \det_{\mathcal{N}(\{1\})}\Bigl(\bigl(\overline{c[i]^*_{n+1}}\bigr)^{(2)} \circ \bigl(q[i]_3 \circ k[i]_3\bigr)^{(2)} \circ \bigl(\overline{c[i]_{n+1}}\bigr)^{(2)}\Bigr) \\
&= \det_{\mathcal{N}(\{1\})}\Bigl(\bigl(\overline{c[i]^*_{n+1}}\bigr)^{(2)}\Bigr) \cdot \det_{\mathcal{N}(\{1\})}\Bigl(\bigl(q[i]_3 \circ k[i]_3\bigr)^{(2)}\Bigr) \cdot \det_{\mathcal{N}(\{1\})}\Bigl(\bigl(\overline{c[i]_{n+1}}\bigr)^{(2)}\Bigr)\,.\end{aligned}$$

Since $1 \le \det_{\mathcal{N}(\{1\})}\Bigl(\bigl(\overline{c[i]^*_{n+1}}\bigr)^{(2)}\Bigr)$ and $1 \le \det_{\mathcal{N}(\{1\})}\Bigl(\bigl(\overline{c[i]_{n+1}}\bigr)^{(2)}\Bigr)$, this implies

$$\det_{\mathcal{N}(\{1\})}\Bigl(\bigl(q[i]_3 \circ k[i]_3\bigr)^{(2)}\Bigr) \quad \le \quad \Bigl(\det_{\mathcal{N}(\{1\})}\bigl(c[i]^{(2)}_{n+1}\bigr)\Bigr)^2\,. \tag{3.38}$$

We derive from (3.36), (3.37) and (3.38)

$$1 \le \det_{\mathcal{N}(\{1\})}\big(u[i]^{(2)}\big) \le \Big(\det_{\mathcal{N}(\{1\})}\big(c[i]_{n+1}^{(2)}\big)\Big)^2. \tag{3.39}$$

We conclude from (3.31), (3.32), (3.33), and (3.39)

$$\Big(\det_{\mathcal{N}(\{1\})}\big(c[i]_{n+1}^{(2)}\big)\Big)^{-2} \le \det_{\mathcal{N}(\{1\})}(\alpha[i]_n) \le \det_{\mathcal{N}(\{1\})}\big(\Delta[i]_n^{(2)}\big). \tag{3.40}$$

We conclude from [25, Lemma 1.18 on page 24 and Lemma 3.15 (4) and (7) on page 129 and 130] (see also the proof of [25, Lemma 3.30 on page 140])

$$\begin{aligned} \ln\big(\det_{\mathcal{N}(\{1\}}(\Delta[i]_n^{(2)})\big) & \\ &= 2\cdot\Big(\ln\big(\det_{\mathcal{N}(\{1\}}(c[i]_n^{(2)})\big) + \ln\big(\det_{\mathcal{N}(\{1\}}(c[i]_{n+1}^{(2)})\big)\Big). \end{aligned} \tag{3.41}$$

Now (3.28) and (3.41) imply for all $n \ge 0$

$$\begin{aligned} \frac{\ln\big(\det_{\mathcal{N}(\{1\}}(\Delta[i]_n)\big)}{[G:G_i]} &\le \Lambda; \\ \frac{\ln\big(\det_{\mathcal{N}(\{1\}}(c[i]_n)^{-2}\big)}{[G:G_i]} &\ge -\Lambda. \end{aligned}$$

This implies together with (3.40) for all $i \in I$ and $n \ge 0$

$$-\Lambda \le \ln\big(\det_{\mathcal{N}(\{1\})}(\alpha[i]_n)\big) \le \Lambda.$$

This finishes the proof of Lemma 3.26. $\square$

3.6. **The maps $\alpha[i]_n$.** In this subsection we deal in a slightly more general context with the maps $\alpha[i]_n$. Notice that in the lemma below the group N is not required to be abelian, the condition about the N-action is on homology with $\mathbb{Q}$-coefficients and the number n is fixed.

Remark 3.42 (Strategy)**.** The proof of the next result will reveal our main strategy and idea. We will consider the quotient $Q = G/N$. We will use the a priori bounds above applied to everything on the Q-level for the system $\{Q_i \mid i \in I\}$, where Q_i is the image of G_i under the projection $G \to Q$. We will show that the values on the G-level and Q level differ in a controlled manner. Finally we use the facts that $[G : G_i] = [N : (N \cap G_i)] \cdot [Q : Q_i]$ and $\lim_{i\to\infty}[N : (N \cap G_i)] = \infty$.

Lemma 3.43. *Let C_* be a based free $\mathbb{Z}G$-chain complex. Fix $n \ge 0$. Assume that there is an infinite normal subgroup $N \subseteq G$ and an index i_0 such that N acts trivially on $\mathbb{Q} \otimes_{\mathbb{Z}} H_n(C[i]_*)$ for every $i \ge i_0$. Then:*

$$\lim_{i\in I}\left|\frac{\ln\big(\det_{\mathcal{N}(\{1\})}(\alpha[i]_n)\big)}{[G:G_i]}\right| = 0.$$

Proof. We can assume without loss of generality $G = G_{i_0}$ and that N operates trivially on $H_n(\mathbb{Q} \otimes_{\mathbb{Z}} C[i]_*)$ for all $i \in I$, since $C[i]_*$ agrees as based free $\mathbb{Z}$-chain complex with $\mathbb{Z}[G_{i_0}/G_i] \otimes_{\mathbb{Z}[G_{i_0}]} C_*$ and $[G : G_i] = [G_{i_0} : G_i] \cdot [G : G_{i_0}]$ holds for $i \ge i_0$ and $N \cap G_{i_0}$ is an infinite normal subgroup of G_{i_o}.

Put $Q = G/N$. Let $\mathrm{pr}\colon G \to Q$ be the projection. Put $Q_i = \mathrm{pr}(G_i)$. Then we obtain an exact sequence of finite groups $1 \to N/(N \cap G_i) \to G/G_i \to Q/Q_i \to 1$. This implies for every $i \in I$

$$[G : G_i] \quad = \quad [N : (N \cap G_i)] \cdot [Q : Q_i]. \tag{3.44}$$

Define the finite based free $\mathbb{Z}Q$-chain complex D_* by $D_* = \mathbb{Z}Q \otimes_{\mathbb{Z}G} C_*$. Define the based free finite $\mathbb{Z}[Q/Q_i]$-chain complex $D[i]_*$ by $D[i]_* = \mathbb{Z}[Q/Q_i] \otimes_{\mathbb{Z}Q} D_* =$

$\mathbb{Z}[Q/Q_i] \otimes_{\mathbb{Z}[G/G_i]} C[i]_*$. We get $D[i]_* = \mathbb{Z} \otimes_{\mathbb{Z}[N/(N\cap G_i)]} C[i]_*$ as based free $\mathbb{Z}$-chain complexes.

Let $\mathrm{pr}[i]_* \colon C[i]_* \to D[i]_*$ be the canonical projection. Let $j[i]_* \colon \big(C[i]_*\big)^{N/(N\cap G_i)} \to D[i]_* = \mathbb{Z} \otimes_{\mathbb{Z}[N/(N\cap G_i)]} C[i]_*$ be the obvious chain map. Let $E[i]_*$ be its cokernel. Each $E[i]_n$ is annihilated by multiplication with $|N/(N\cap G_i)|$. Since we obtain a short exact sequence of $\mathbb{Z}$-chain complexes $0 \to \big(C[i]_*\big)^{N/(N\cap G_i)} \xrightarrow{j[i]_*} D[i]_* \to E[i]_* \to 0$, we conclude by considering the associated long homology sequence, that the kernel and the cokernel of the map $H_n(j[i]_*) \colon H_n\big((C[i]_*)^{N/(N\cap G_i)}\big) \to H_n\big(D[i]_*\big)$ is annihilated by multiplication with $|N/(N\cap G_i)|$. Since $N/(N\cap G_i)$ acts trivially on $\mathbb{Q}\otimes_{\mathbb{Z}} H_n\big(C[i]_*\big)$, the canonical map $\mathbb{Q}\otimes_{\mathbb{Z}} H_n\big((C[i]_*)^{N/(N\cap G_i)}\big) \to \mathbb{Q}\otimes_{\mathbb{Z}} H_n(C[i]_*)$ is bijective. Hence the map $H_n(\mathrm{pr}[i]_*)_f \colon H_n(C[i]_*)_f \to H_n(D[i]_*)_f$ is injective and its cokernel is annihilated by multiplication with $|N/(N\cap G_i)|$. This implies together with Lemma 2.4 applied to $H_n(\mathrm{pr}[i]_*)_f$ viewed as 1-dimensional finite free $\mathbb{Z}$-chain complex

$$\begin{aligned} 1 \le \det_{\mathcal{N}(\{1\})}\big(\big(H_n(\mathrm{pr}[i]_*)_f\big)^{(2)}\big) &= \big|\mathrm{tors}\big(\mathrm{coker}(H_n(\mathrm{pr}[i]_*)_f)\big)\big| \\ &\le |N/(N\cap G_i)|^{\mathrm{rk}_{\mathbb{Z}}(H_n(D[i]_*)_f)}. \end{aligned}$$

This together with (3.44) shows

$$\begin{aligned} 0 &\le \frac{\ln\Big(\det_{\mathcal{N}(\{1\})}\big(\big(H_n(\mathrm{pr}[i]_*)_f\big)^{(2)}\big)\Big)}{[G:G_i]} \\ &\le \frac{\ln\big([N:(N\cap G_i)]\big)}{[N:(N\cap G_i)]} \cdot \frac{\mathrm{rk}_{\mathbb{Z}}(H_n(D[i]_*)_f)}{[Q:Q_i]} \\ &\le \frac{\ln\big([N:(N\cap G_i)]\big)}{[N:(N\cap G_i)]} \cdot \frac{\mathrm{rk}_{\mathbb{Z}}(D[i]_n)}{[Q:Q_i]} \\ &\le \frac{\ln\big([N:(N\cap G_i)]\big)}{[N:(N\cap G_i)]} \cdot \mathrm{rk}_{\mathbb{Z}Q}(D_n). \end{aligned}$$

Since $\lim_{i\in I}[N:(N\cap G_i)] = \infty$ and hence $\lim_{i\in I}\frac{\ln\left([N:(N\cap G_i)]\right)}{[N:(N\cap G_i)]} = 0$, we conclude

$$\lim_{i\in I} \frac{\ln\Big(\det_{\mathcal{N}(\{1\})}\big(\big(H_n(\mathrm{pr}[i]_*)_f\big)^{(2)}\big)\Big)}{[G:G_i]} = 0. \tag{3.45}$$

The following diagram commutes

$$\begin{tikzcd} \big(C[i]_n^{(2)}\big)^{N/(N\cap G_i)} \arrow[r, "\left(\Delta[i]_n^{(2)}\right)^{N/(N\cap G_i)}"] \arrow[d, "k[i]_n"] & \big(C[i]_n^{(2)}\big)^{N/(N\cap G_i)} \arrow[d, "k[i]_n"] \\ C[i]_n^{(2)} \arrow[r, "\Delta[i]_n^{(2)}"] \arrow[d, "\mathrm{pr}[i]_n^{(2)}"] & C[i]_n^{(2)} \arrow[d, "\mathrm{pr}[i]_n^{(2)}"] \\ D[i]_n^{(2)} \arrow[r, "\Delta[i,D]_n^{(2)}"] & D[i]_n^{(2)} \end{tikzcd}$$

where $k[i]_n$ is the inclusion, $\Delta[i]_n^{(2)}$ is the nth Laplacian of $C[i]_*^{(2)}$ and $\Delta[i,D]_n^{(2)}$ is the nth Laplacian of $D[i]_*^{(2)}$.

Recall that we equip $\mathbb{C}[N/(N\cap G_i)]$ and $\mathbb{C}\otimes_{\mathbb{C}[N/(N\cap G_i)]}\mathbb{C}[N/(N\cap G_i)]$ with the Hilbert space structure coming from the obvious $\mathbb{C}$-basis. Hence $N/(N\cap G_i)$ and $\{1\otimes e\}$ are Hilbert basis for $\mathbb{C}[N/(N\cap G_i)]$ and $\mathbb{C}\otimes_{\mathbb{C}[N/(N\cap G_i)]}\mathbb{C}[N/(N\cap G_i)]$.

We equip $\mathbb{C}[N/(N \cap G_i)]^{N/(N\cap G_i)} \subseteq \mathbb{C}[N/(N \cap G_i)]$ with the sub Hilbert space structure. Let $N = \sum_{\overline{k} \in N/(N\cap G_i)} \overline{k} \in \mathbb{C}[N/(N \cap G_i)]$ be the norm element. Then $\{N/\sqrt{[N : (N \cap G_i)]}\}$ is a Hilbert basis of the Hilbert space $\mathbb{C}[N/(N\cap G_i)]^{N/(N\cap G_i)}$. The obvious composite

$$\mathbb{C}[N/(N \cap G_i)]^{N/(N\cap G_i)} \to \mathbb{C}[N/(N \cap G_i)] \to \mathbb{C} \otimes_{\mathbb{C}[N/(N\cap G_i)]} \mathbb{C}[N/(N \cap G_i)]$$

sends $N/\sqrt{[N : (N \cap G_i)]}$ to an element of norm $\sqrt{[N : (N \cap G_i)]}$. This implies that the composite $\operatorname{pr}[i]_n^{(2)} \circ k[i]_n \colon \big(C[i]_n^{(2)}\big)^{N/(N\cap G_i)} \to D[i]_n^{(2)}$ satisfies

$$||x||_{L^2} \le \big|\big|\operatorname{pr}[i]_n^{(2)} \circ j[i]_n(x)\big|\big|_{L^2} \quad \text{for } x \in \big(C[i]_n^{(2)}\big)^{N/(N\cap G_i)}.$$

We conclude that the induced map

$$\operatorname{pr}[i]_n^{(2)} \circ k[i]_n\big|_{\ker((\Delta[i]_n^{(2)})^{N/(N\cap G_i)})} \colon \ker\big(\big(\Delta[i]_n^{(2)}\big)^{N/(N\cap G_i)}\big) \to \ker\big(\Delta[i,D]_n^{(2)}\big)$$

satisfy the corresponding inequality. Notice that $\big(\Delta[i]_n^{(2)}\big)^{N/(N\cap G_i)}$ can be viewed as the nth Laplacian of the Hilbert chain complex $\big(C[i]_n^{(2)}\big)^{N/(N\cap G_i)}$. Hence the map

$$H_n^{(2)}\big(\operatorname{pr}[i]_*^{(2)} \circ k[i]_*\big) \colon H_n^{(2)}\left(\big(C[i]_*^{(2)}\big)^{N/(N\cap G_i)}\right) \to H_n^{(2)}\big(D[i]_*^{(2)}\big)$$

satisfies

$$||y||_{L^2} \le \big|\big|H_n^{(2)}\big(\operatorname{pr}[i]_*^{(2)} \circ k[i]_*\big)(y)\big|\big|_{L^2} \quad \text{for } y \in H_n^{(2)}\left(\big(C[i]_*^{(2)}\big)^{N/(N\cap G_i)}\right).$$

since the obvious maps $\ker\big(\big(\Delta[i]_n^{(2)}\big)^{N/(N\cap G_i)}\big) \to H_n^{(2)}\left(\big(C[i]_*^{(2)}\big)^{N/(N\cap G_i)}\right)$ and $\ker\big(\Delta[i,D]_n^{(2)}\big) \to H_n^{(2)}\big(D[i]_*^{(2)}\big)$ are isometries. This implies that the norm and hence also the Fuglede-Kadison determinant of the inverse of $H_n^{(2)}\big(\operatorname{pr}[i]_*^{(2)} \circ k[i]_*\big)$ is bounded by 1. Since both $H_n^{(2)}(\operatorname{pr}[i]_*^{(2)})$ and $H_n^{(2)}(k[i]_*)$ are invertible, we conclude from [25, Theorem 3.14 (1) on page 128]

$$1 \quad \le \quad \det_{\mathcal{N}(\{1\})}\big(H_n^{(2)}(\operatorname{pr}[i]_*^{(2)})\big) \cdot \det_{\mathcal{N}(\{1\})}\big(H_n^{(2)}(k[i]_*)\big). \tag{3.46}$$

The projection $\mathbb{C}[N/(N \cap G_i)] \to \mathbb{C} \otimes_{\mathbb{C}[N/(N\cap G_i)} \mathbb{C}[N/(N \cap G_i)]$ sends each element in $N/(N \cap G_i)$ to an element of norm 1. Hence the operator norm of $\operatorname{pr}[i]_n^{(2)} \colon C[i]_n^{(2)} \to D[i]_n^{(2)}$ is bounded by $[N : (N \cap G_i]$. This implies that the operator norm of $H_n^{(2)}(\operatorname{pr}[i|_*^{(2)}) \colon H_n^{(2)}(C[i]_*) \to H_n(D[i]_*)$ is bounded by $[N : (N \cap G_i]$. We conclude

$$\det_{\mathcal{N}(\{1\})}\big(H_n^{(2)}(\operatorname{pr}[i|_*^{(2)})\big) \quad \le \quad [N : (N \cap G_i]^{\dim_{\mathbb{C}}(H_n^{(2)}(D[i]_*^{(2)})}. \tag{3.47}$$

Since $k[i]_*$ is an isometric embedding, the norm of the induced map $H_n^{(2)}(k[i]_*)$ is bounded by 1. Hence

$$\det_{\mathcal{N}(\{1\})}\big(H_n^{(2)}(k[i|_*)\big) \quad \le \quad 1. \tag{3.48}$$

Putting (3.46), (3.47) and (3.48) together yields

$$1 \quad \le \quad \det_{\mathcal{N}(\{1\})}\big(H_n^{(2)}\big(\operatorname{pr}[i]_n\big) \le [N : (N \cap G_i)]^{\dim_{\mathbb{C}}(H_n^{(2)}(D[i]_*^{(2)})}.$$

The same argument as in the end of the proof of (3.45) shows

$$\lim_{i \in I} \frac{\ln\Big(\det_{\mathcal{N}(\{1\}}\big(H_n^{(2)}(\operatorname{pr}[i]_*^{(2)})\big)\Big)}{[G : G_i]} \quad = \quad 0. \tag{3.49}$$

We obtain from Lemma 3.26 (4) applied to Q, $\{Q_i \mid i \in I\}$ and D_* that there exists a constant Λ independent of $i \in I$ such that for all $i \in I$

$$\frac{\ln\big(\det_{\mathcal{N}(\{1\}}(\alpha[i,D]_n)\big)}{[Q : Q_i]} \quad \le \quad \Lambda. \tag{3.50}$$

Since $\lim_{i\in}[N : (N \cap G_i)] = \infty$, we derive from (3.44) and (3.50)

$$\frac{\ln\left(\det_{\mathcal{N}(\{1\}}(\alpha[i,D]_n)\right)}{[G:G_i]} \quad = \quad 0. \tag{3.51}$$

The following diagram of isomorphisms of $\mathcal{N}(\{1\})$-Hilbert modules commutes

$$\begin{array}{ccc}
\big(H_n(C[i]_*)_f\big)^{(2)} & \xrightarrow{\alpha[i]_n} & H_n^{(2)}\big(C[i]_*^{(2)}\big) \\
{\scriptstyle \left(H_n(\mathrm{pr}[i]_*)_f\right)^{(2)}}\Big\downarrow & & \Big\downarrow{\scriptstyle H_n^{(2)}\left(\mathrm{pr}[i]_*^{(2)}\right)} \\
\big(H_n(D[i]_*)_f\big)^{(2)} & \xrightarrow[\alpha[i,D]_n]{} & H_n^{(2)}\big(D[i]_*^{(2)}\big)
\end{array}$$

where $\alpha[i]_n$ and $\alpha[i,D]_n$ respectively is the map associated to the finite based free $\mathbb{Z}$-chain complex $C[i]_*$ and $D[i]_*$ respectively in (2.3). We conclude from [25, Theorem 3.14 (1) on page 128]

$$\begin{aligned}\ln\left(\det_{\mathcal{N}(\{1\}}(\alpha[i]_n)\right) = \ln\left(\det_{\mathcal{N}(\{1\}}(\alpha[i,D]_n)\right) + \ln\left(\det_{\mathcal{N}(\{1\}}\left(\left(H_n(\mathrm{pr}[i]_*)_f\right)^{(2)}\right)\right) \\ - \ln\left(\det_{\mathcal{N}(\{1\}}\left(H_n^{(2)}(\mathrm{pr}[i]_*^{(2)})\right)\right).\end{aligned}$$

This together with (3.45), (3.49), and (3.51) implies

$$\lim_{i\in I}\left|\frac{\ln\left(\det_{\mathcal{N}(\{1\})}(\alpha[i]_n)\right)}{[G:G_i]}\right| \quad = \quad 0.$$

This finishes the proof Lemma 3.43. $\square$

Example 3.52. Let X be a connected CW-complex with fundamental group $G = \pi_1(X)$. Since G acts trivially on $H_0(G_i\backslash\widetilde{X})$ for all $i \in I$, Lemma 3.43 implies

$$\lim_{i\in I}\left|\frac{\ln\left(\det_{\mathcal{N}(\{1\})}(\alpha[i]_0)\right)}{[G:G_i]}\right| = 0.$$

Let M be a closed manifold of dimension d with fundamental group $G = \pi_1(X)$. Let G_0 be the subgroup of index 1 or 2 of those elements in G which act orientation preserving on $\widetilde{M}$. Then G_0 acts trivially on $H_d(G_i\backslash\widetilde{M})$ for all $i \in I$. We conclude from Lemma 3.43

$$\lim_{i\in I}\left|\frac{\ln\left(\det_{\mathcal{N}(\{1\})}(\alpha[i]_d)\right)}{[G:G_i]}\right| = 0.$$

If we additionally assume that $G_i\backslash\widetilde{M}$ is a rational homology sphere for all $i \in I$, then Lemma 2.4 implies

$$\lim_{i\in I}\frac{\rho^{(2)}\big(G_i\backslash\widetilde{\widetilde{M}}\big)}{[G:G_i]} - \frac{\rho^{\mathbb{Z}}\big(G_i\backslash\widetilde{M}\big)}{[G:G_i]} = 0.$$

3.7. **Proof of a chain complex version.** In this subsection we prove

Proposition 3.53. *Let C_* be a finite based free $\mathbb{Z}G$-chain complex. Consider a natural number d. Assume that there is an infinite abelian normal subgroup $A \subseteq G$, an index i_0 and a natural number r such that $H_n(C[i]_*)$ is a nilpotent $\mathbb{Z}[A/(A\cap G_i)]$-module of filtration length $\leq r$ for every $i \geq i_0$ and $n \leq d$. Then:*

1. *We get for all $n \leq d$*

$$\lim_{i\in I}\frac{\mathrm{d}(H_n(C_*[i]))}{[G:G_i]} = 0;$$

(2) We get for all $n \le d$

$$\lim_{i \in I} \frac{\dim_K\bigl(H_n(C_*[i];K)\bigr)}{[G:G_i]} = 0;$$

(3) We get for all $n \le d$

$$\lim_{i \in I} \frac{\ln(|\operatorname{tors}(H_n(C_*[i]))|)}{[G:G_i]} = 0;$$

(4) We get for all $n \le d$

$$\lim_{i \in I} \frac{\bigl|\ln(\det_{\mathcal{N}(\{1\})}(\alpha[i]_n))\bigr|}{[G:G_i]} = 0;$$

(5) We get

$$\lim_{i \in I} \rho^{(2)}\bigl(C[i]_*;\mathcal{N}(G/G_i)\bigr) = \lim_{i \in I} \frac{\rho^{\mathbb{Z}}(C[i]_*)}{[G:G_i]} = 0.$$

Proof. We can assume without loss of generality $G = G_{i_0}$ and that $H_n(C[i]_*)$ is a nilpotent $\mathbb{Z}[A/(A \cap G_i)]$-module of filtration length $\le r$ for all $i \in I$ and $p \ge 0$, since $C[i]_*$ agrees as based free $\mathbb{Z}$-chain complex with $\mathbb{Z} \otimes_{\mathbb{Z}[G_i]} C_*$ and $[G:G_i] = [G_{i_0} : G_i] \cdot [G : G_{i_0}]$ holds for $i \ge i_0$, and $A \cap G_{i_0}$ is an infinite normal subgroup of G_{i_0}.

Put $Q = G/A$. Let $\operatorname{pr}\colon G \to Q$ be the projection. Put $Q_i = \operatorname{pr}(G_i)$. Then we obtain an exact sequence of finite groups $1 \to A/(A \cap G_i) \to G/G_i \to Q/Q_i \to 1$. This implies for every $i \in I$

$$[G : G_i] \quad = \quad [A : (A \cap G_i)] \cdot [Q : Q_i]. \tag{3.54}$$

Define the finite based free $\mathbb{Z}Q$-chain complex D_* by $D_* = \mathbb{Z}Q \otimes_{\mathbb{Z}G} C_*$. Define the based free finite $\mathbb{Z}[Q/Q_i]$-chain complex $D[i]_*$ by $D[i]_* = \mathbb{Z}[Q/Q_i] \otimes_{\mathbb{Z}Q} D_* = \mathbb{Z}[Q/Q_i] \otimes_{\mathbb{Z}[G/G_i]} C[i]_*$. We get $D[i]_* = \mathbb{Z} \otimes_{\mathbb{Z}[A/(A\cap G_i)]} C[i]_*$ as based free $\mathbb{Z}$-chain complexes.

Let $\operatorname{pr}[i]_*\colon C[i]_* \to D[i]_*$ be the canonical projection. We conclude from Proposition 3.13 (applied to the finite abelian group $A/(A\cap G_i)$) that there are functions

$$C_0, C_1, D_0, D_1\colon \bigl\{(r,n,p) \in \mathbb{N}^3 \mid p \le n\bigr\} \quad \to \quad \mathbb{R},$$

such that the following is true for every $n \ge 0$ and $i \in I$

$$\mathrm{d}(H_n(C[i]_*)) \quad \le \quad \sum_{p=0}^{n} C_0(r,n,p) \cdot \mathrm{d}\bigl(A/(A \cap G_i)\bigr)^{C_1(r,n,p)} \cdot \mathrm{d}\bigl(H_p(D[i]_*)\bigr); \tag{3.55}$$

$$\ln\bigl(\bigl|\ker(H_n(\operatorname{pr}[i]_*))\bigr|\bigr) \quad \le \quad \sum_{p=0}^{n} D_0(r,n,p) \cdot \ln\bigl(|A/(A \cap G_i)|\bigr) \cdot \mathrm{d}\bigl(A/(A \cap G_i)\bigr)^{D_1(r,n,p)} \cdot \mathrm{d}\bigl(H_p(D[i]_*)\bigr); \tag{3.56}$$

$$\ln\bigl(\bigl|\operatorname{coker}(H_n(\operatorname{pr}[i]_*))\bigr|\bigr) \quad \le \quad \sum_{p=0}^{n} D_0(r,n,p) \cdot \ln\bigl(|A/(A \cap G_i)|\bigr) \cdot \mathrm{d}\bigl(A/(A \cap G_i)\bigr)^{D_1(r,n,p)} \cdot \mathrm{d}\bigl(H_p(D[i]_*)\bigr). \tag{3.57}$$

We obtain from Lemma 3.26 applied to Q, $\{Q_i \mid i \in I\}$ and D_* that there exists a constant Λ independent of $i \in I$ such that for all $i \in I$ and $p \geq 0$ we get

$$\frac{\mathrm{d}\big(H_p(D[i]_*)\big)}{[Q:Q_i]} \quad \leq \quad \Lambda; \tag{3.58}$$

$$\frac{\ln\big(\mathrm{tors}(H_p(D[i]_*))\big)}{[Q:Q_i]} \quad \leq \quad \Lambda. \tag{3.59}$$

We conclude from (3.54), (3.55), (3.56), (3.57), (3.58), and (3.59)

$$\frac{\mathrm{d}(H_n(C[i]_*))}{[G:G_i]} \quad \leq \quad \sum_{p=0}^{n} \Lambda \cdot C_0(r,n,p) \cdot \frac{\mathrm{d}\big(A/(A\cap G_i)\big)^{C_1(r,n,p)}}{[A:(A\cap G_i)]}; \tag{3.60}$$

$$\begin{aligned} &\frac{\ln\left(\left|\ker(H_n(\mathrm{pr}[i]_*))\right|\right)}{[G:G_i]} \\ &\qquad \leq \sum_{p=0}^{n} \Lambda \cdot D_0(r,n,p) \cdot \frac{\ln\big(|A/(A\cap G_i)|\big) \cdot \mathrm{d}\big(A/(A\cap G_i)\big)^{D_1(r,n,p)}}{[A:(A\cap G_i)]}; \end{aligned} \tag{3.61}$$

$$\begin{aligned} &\frac{\ln\left(\left|\mathrm{coker}(H_n(\mathrm{pr}[i]_*))\right|\right)}{[G:G_i]} \\ &\qquad \leq \sum_{p=0}^{n} \Lambda \cdot D_0(r,n,p) \cdot \frac{\ln\big(|A/(A\cap G_i)|\big) \cdot \mathrm{d}\big(A/(A\cap G_i)\big)^{D_1(r,n,p)}}{[A:(A\cap G_i)]}. \end{aligned} \tag{3.62}$$

One shows for every natural number m by induction over m using L'Hospital's rule

$$\lim_{x\to\infty} \frac{\ln(x)^m}{x} \quad = \quad 0. \tag{3.63}$$

We conclude from Lemma 2.13 (3)

$$\mathrm{d}\big(A/(A\cap G_i)\big) \quad \leq \quad \frac{\ln\big([A:(A\cap G_i)]\big)}{\ln(2)}. \tag{3.64}$$

Since A is infinite and $\bigcap_{i\in I} G_i = \{1\}$, we have

$$\lim_{i\to\infty} [A:(A\cap G_i)] \quad = \quad \infty. \tag{3.65}$$

We conclude from (3.63), (3.64) and (3.65)

$$\lim_{i\to\infty} \frac{\mathrm{d}\big(A/(A\cap G_i)\big)^{C_1(r,n,p)}}{[A:(A\cap G_i)]} \quad = \quad 0; \tag{3.66}$$

$$\lim_{i\to\infty} \frac{\ln\big(|A/(A\cap G_i)|\big) \cdot \mathrm{d}\big(A/(A\cap G_i)\big)^{C_1(r,n,p)}}{[A:(A\cap G_i)]} \quad = \quad 0. \tag{3.67}$$

Now (3.60), (3.61),(3.62), (3.66) and (3.67) imply

$$\lim_{i\in I} \frac{\mathrm{d}(H_n(C[i]_*))}{[G:G_i]} \quad = \quad 0; \tag{3.68}$$

$$\lim_{i\in I} \frac{\ln\left(\left|\ker(H_n(\mathrm{pr}[i]_*))\right|\right)}{[G:G_i]} \quad = \quad 0; \tag{3.69}$$

$$\lim_{i\in I} \frac{\ln\left(\left|\mathrm{coker}(H_n(\mathrm{pr}[i]_*))\right|\right)}{[G:G_i]} \quad = \quad 0. \tag{3.70}$$

Now assertion (1) follows from (3.68). Since by the universal coefficient theorem we have

$$\dim_K\big(H_n(C[i]_*;K)\big) \leq \mathrm{d}(H_n(C[i]_*) + \mathrm{d}(H_{n-1}(C[i]_*)$$

assertion (2) follows from assertion (1).

We conclude from (3.69) and (3.70)

$$\lim_{i\in I}\left|\frac{\ln\left(\left|\operatorname{tors}(H_n(C[i]_*))\right|\right)}{[G:G_i]}-\frac{\ln\left(\left|\operatorname{tors}(H_n(D[i]_*))\right|\right)}{[G:G_i]}\right| = 0. \tag{3.71}$$

We conclude from (3.54) and (3.59)

$$\lim_{i\in I}\frac{\ln\left(\left|\operatorname{tors}(H_n(D[i]_*))\right|\right)}{[G:G_i]} = 0. \tag{3.72}$$

We derive from (3.71) and (3.72)

$$\lim_{i\in I}\frac{\ln\left(\left|\operatorname{tors}(H_n(C[i]_*))\right|\right)}{[G:G_i]} = 0. \tag{3.73}$$

Now assertion (3) follows from (3.73).

Assertion (4) is a special case of Lemma 3.43 since $A/(A\cap G_i)$ is finite and hence for a nilpotent $\mathbb{Z}[A/(A\cap G_i)]$-module M the $A/(A\cap G_i)$-action on $\mathbb{Q}\otimes_{\mathbb{Z}} M$ is trivial.

Assertion (5) follows from assertions (3) and (4) together with Lemma 2.4.

This finishes the proof of Proposition 3.53. □

3.8. **Proof of a CW-complex version.** In this subsection we prove

Theorem 3.74 (CW-version)**.** *Let X be a connected CW-complex. Consider an epimorphism $\phi\colon \pi_1(X)\to G$. Let $p\colon \overline{X}\to X$ be the associated G-covering. Consider a natural number d. Assume that there is an infinite abelian normal subgroup $A\subseteq G$, an index i_0 and a natural number r such that $H_n(G_i\backslash\overline{X})$ is a nilpotent $\mathbb{Z}[A/(A\cap G_i)]$-module of filtration length $\le r$ for every $i\ge i_0$ and $n\le d$ and that the $(d+1)$-skeleton of X is finite. Then:*

1. *We get for all $n\le d$*
$$\lim_{i\in I}\frac{\operatorname{d}\bigl(H_n(G_i\backslash\overline{X})\bigr)}{[G:G_i]}=0;$$
2. *We get for all $n\le d$*
$$\lim_{i\in I}\frac{\ln\left(\left|\operatorname{tors}\bigl(H_n(G_i\backslash\overline{X})\bigr)\right|\right)}{[G:G_i]}=0;$$
3. *We get for all $n\le d$*
$$0=b_n^{(2)}\bigl(\overline{X};\mathcal{N}(G)\bigr)=\lim_{i\to\infty}\frac{b_n(G_i\backslash X;K)}{[G:G_i]}=0;$$
4. *Suppose that X is a finite connected CW-complexes. Then*
$$0=\lim_{i\in I}\rho^{(2)}\bigl(G_i\backslash\overline{X};\mathcal{N}(G/G_i)\bigr)=\lim_{i\in I}\frac{\rho^{\mathbb{Z}}\bigl(G_i\backslash\overline{X}\bigr)}{[G:G_i]}.$$

Proof. Assertions (1), (2) and (3) follows from Proposition 3.53 applied to the cellular $\mathbb{Z}G$-chain complex of $\overline{X}$ truncated in dimensions greater or equal to $(d+2)$, since this truncation does does not change the homology in dimension $\le d$, and from [24, Theorem 0.1]. Assertion (4) follows from Proposition 3.53 applied to the cellular $\mathbb{Z}G$-chain complex of $\overline{X}$ itself. □

Remark 3.75. Notice that in the situation of Theorem 3.74 we do *not* claim $\lim_{i\in I}\frac{\ln(\det(c_n[i]))}{[G:G_i]}=0$. A counterexample comes from $S^1\times Y$ for a simply connected CW-complex Y whose homology contains no torsion but whose differentials do not all have trivial Fuglede-Kadison determinant.

3.9. **Finishing the proof of Theorem 1.1.** In this section we finish the proof of Theorem 1.1. For this purpose we will need

Lemma 3.76. *Consider the situation of Theorem 1.1. Let $\widehat{G} \subseteq G$ be the image of $\phi\colon \pi_1(X) \to G$. Let $\widehat{\phi}\colon \pi_1(X) \to \widehat{G}$ be the epimorphism induced by ϕ and let $i\colon \widehat{G} \to G$ be the inclusion. Put $\widehat{G}_i := \widehat{G} \cap G_i$. Suppose that Theorem 1.1 holds for $\widehat{\phi}$ and the directed system $\{\widehat{G}_i \mid i \in I\}$.*

Then Theorem 1.1 holds for also for ϕ and the directed system $\{G_i \mid i \in I\}$.

Proof. Recall that $\overline{X} \to X$ is the G-covering associated to $\phi\colon \pi_1(X) \to G$. Let $\widehat{X} \to X$ be the $\widehat{G}$-covering associated to $\widehat{\phi}\colon \pi_1(X) \to \widehat{G}$. Then there is a G-homeomorphism $G \times_{\widehat{G}} \widehat{X} \xrightarrow{\cong} \overline{X}$. There is an obvious G/G_i-homeomorphism

$$G_i\backslash \left(G \times_{\widehat{G}} \widehat{X}\right) \xrightarrow{\cong} G/G_i \times_{\widehat{G}/\widehat{G}_i} \widehat{G}_i\backslash\widehat{X}.$$

Hence we obtain a G/G_i-homeomorphism

$$G_i\backslash\overline{X} \quad \xrightarrow{\cong} \quad G/G_i \times_{\widehat{G}/\widehat{G}_i} \widehat{G}_i\backslash\widehat{X}.$$

Since $\widehat{G}/\widehat{G}_i$ is a subgroup of G/G_i, this yields a homeomorphism

$$G_i\backslash\overline{X} \quad \xrightarrow{\cong} \quad \coprod_{i=1}^{[G/G_i:\widehat{G}/\widehat{G}_i]} \widehat{G}_i\backslash\widehat{X}.$$

This implies

$$\begin{aligned}
(3.77) \quad \frac{\ln\left(\left|\operatorname{tors}(H_n(G_i\backslash\overline{X}))\right|\right)}{[G:G_i]} &= \frac{[G/G_i:\widehat{G}/\widehat{G}_i]\cdot\ln\left(\left|\operatorname{tors}(H_n(\widehat{G}_i\backslash\widehat{X}))\right|\right)}{[G:G_i]} \\
&= \frac{\ln\left(\left|\operatorname{tors}(H_n(\widehat{G}_i\backslash\widehat{X}))\right|\right)}{[\widehat{G}:\widehat{G}_i]}.
\end{aligned}$$

and analogously using Lemma 2.13 (4)

$$(3.78) \qquad \frac{b_n(G_i\backslash\overline{X};K)}{[G:G_i]} = \frac{b_n(\widehat{G}_i\backslash\widehat{X};K)}{[\widehat{G}:\widehat{G}_i]};$$

$$(3.79) \qquad \frac{\operatorname{d}\left(H_n(G_i\backslash\overline{X})\right)}{[G:G_i]} \leq \frac{\operatorname{d}\left(H_n(\widehat{G}_i\backslash\widehat{X})\right)}{[\widehat{G}:\widehat{G}_i]}.$$

We conclude from (3.77)

$$(3.80) \qquad \frac{\rho^{\mathbb{Z}}\left(G_i\backslash\overline{X}\right)}{[G:G_i]} = \frac{\rho^{\mathbb{Z}}\left(\widehat{G}_i\backslash\widehat{X}\right)}{[\widehat{G}:\widehat{G}_i]}.$$

We conclude from [25, Theorem 1.35 (10) on page 38 and Theorem 3.93 (6) on page 162]

$$(3.81) \qquad b_n^{(2)}\left(\overline{X},\mathcal{N}(G)\right) = b_n^{(2)}\left(\widehat{X},\mathcal{N}(\widehat{G})\right);$$

$$(3.82) \qquad \rho^{(2)}\left(\overline{X},\mathcal{N}(G)\right) = \rho_n^{(2)}\left(\widehat{X},\mathcal{N}(\widehat{G})\right),$$

$$(3.83) \qquad \rho^{(2)}\left(G_i\backslash\overline{X},\mathcal{N}(G/G_i)\right) = \rho_n^{(2)}\left(H_i\backslash\widehat{X},\mathcal{N}(\widehat{G}/\widehat{G}_i)\right).$$

Now Lemma 3.76 follows from (3.77), (3.78), (3.79), (3.80), (3.81), (3.82), and (3.83). □

Proof of Theorem 1.1. Because of Lemma 3.76 we can assume in the sequel that $\phi\colon \pi_1(X) \to G$ is surjective.

(1), (2), (3), and (4) Let A be the image of $G_1(F)$ under the composite $\phi \circ$

$\pi_1(j)\colon \pi_1(F) \to \pi_1(X) \to G$. We apply for $i \in I$ Lemma 3.24 to the fibration $F \to p^{-1}(B_{d+1}) \to B_{d+1}$ and the map

$$\phi_i\colon \pi_1(p^{-1}(B_{d+1})) \xrightarrow{\pi_1(i_{d+1})} \pi_1(X) \xrightarrow{\phi} G \xrightarrow{\mathrm{pr}_i} G/G_i$$

where $i_{d+1}\colon p^{-1}(B_{d+1}) \to X$ is the inclusion and pr_i is the projection. Since i_{d+1} is $(d+1)$-connected, we conclude for $n \le d$ that the $\mathbb{Z}[A/(A \cap G_i]$-module $H_n(G_i\backslash X)$ is nilpotent of filtration length $(d+2)$ for $n \le d$. Now we apply Theorem 3.74 taking $r = d+2$.

(5) In the case that $\pi_1(j)\colon \pi_1(F) \to \pi_1(X)$ is injective, $G = \pi_1(X)$, and $\phi\colon \pi_1(X) \to G$ is the identity, the claim follows from [25, Theorem 3.100 on page 166] provided that that $b_n^{(2)}\big(\widetilde{F}; \mathcal{N}(\pi_1(F))\big) = 0$ holds for $n \ge 0$ and that $\rho^{(2)}\big(\widetilde{F}; \mathcal{N}(\pi_1(F))\big) = 0$ is valid. The proof carries directly over to the general case provided that $b_n^{(2)}\big(\overline{F}; \mathcal{N}(H)\big) = 0$ for $n \ge 0$ and $\rho^{(2)}\big(\overline{F}; \mathcal{N}(H)\big) = 0$ holds for the image H of $\phi \circ \pi_1(j)\colon \pi_1(F) \to \pi_1(X) \to G$. We conclude $b_n^{(2)}\big(\overline{F}; \mathcal{N}(H)\big) = 0$ for $n \ge 0$ from assertion (3) applied to the case of the fibration $F \to F \to \{\bullet\}$ and the epimorphism $\pi_1(F) \to H$. We have $\rho^{(2)}\big(\overline{F}; \mathcal{N}(H)\big) = 0$ by assumption. This finishes the proof of Theorem 1.1. □

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